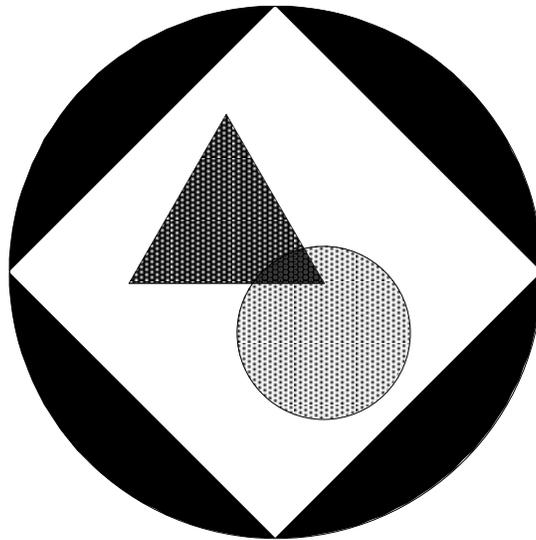


# Professional Development Workshop Guide

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## The Geometer's Sketchpad<sup>®</sup> Workshop Guide



**Key Curriculum Press**  
Innovators in Mathematics Education

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This booklet is designed as a handout for use in a workshop on The Geometer's Sketchpad or for use—in conjunction with the Instructor's Evaluation Edition of The Geometer's Sketchpad—in evaluating the program for possible purchase.

**Note: The Instructor's Evaluation Edition of The Geometer's Sketchpad will expire after 60 days, at which point it will cease to operate on the computer on which it was installed. Reinstallation of the software will not extend the period of operability. Workshop leaders should use the fully functional version of Sketchpad to avoid problems with expiration.**

## To the Workshop Participant

The tours in this booklet will introduce you to many of Sketchpad's features and to methods for using Sketchpad in the classroom. It's easy to retrace your steps, so feel free to explore on your own. Ask for help when you get stuck and offer to help others when you can.

At the end of the workshop, you may take this booklet with you and use it with the Instructor's Evaluation Edition of Sketchpad to explore and evaluate Sketchpad. (You can download the Instructor's Evaluation Edition of Sketchpad from the web at <http://www.keypress.com/sketchpad>.) But remember that the Instructor's Evaluation Edition of Sketchpad and this *Workshop Guide* are for teacher education and evaluation only, and are not for classroom or other extended use.

### How Is the Instructor's Evaluation Edition Different from the "Real Thing"?

The Instructor's Evaluation Edition of Sketchpad does everything the full version does with a few important exceptions: You can't print, save, or export your work, and the extensive help system is missing. The full version of Sketchpad includes extensive documentation, teacher support materials, and the help system. Documentation includes a *Learning Guide*, a 252-page *Reference Manual*, and *Teaching Mathematics*, which provides tips for using Sketchpad in the classroom and 25 reproducible sample classroom activities.

## To the Workshop Presenter

A successful workshop balances providing guidance with allowing participants to explore on their own. Allotting specific amounts of time for small chunks of each tour works best, so time intervals are marked in the tours. Use a watch or a kitchen timer, and interrupt participants at the end of each section. Ask for questions and demonstrate what they've just done on the overhead, if one is available. Keep things moving along and ensure that nobody spends too long stuck on a particular concept.

In addition to teaching the participants about Sketchpad, you should also be teaching and modeling good "lab etiquette." When participants have a question, they should ask their neighbors—whether it's a question about math or about Sketchpad—before asking you. When answering participants' questions, avoid the temptation to take over their computer! They'll learn by doing it themselves, not by watching you do it. (Participants should abide by the same rule.)

Make sure that the participants know the agenda for the day and that they are aware of the range of activities and resources available to them. Encourage them to explore on their own when they've finished a scheduled activity, but to choose materials that won't be specifically covered during the workshop.

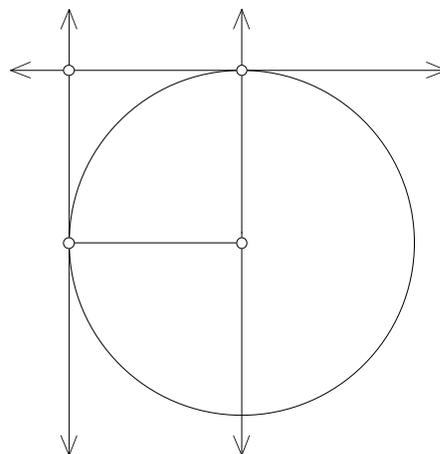
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## Tour 1: Constructing a Square

In this tour, you'll learn about Sketchpad's basic tools and get a first taste of constructing a geometric figure—a square.

### What You Will Learn

- How to construct segments and circles
- How to select and drag objects
- How to construct lines that are perpendicular or parallel to other lines
- How to construct points at the intersection of two objects
- How to save Sketchpad documents
- How to use the **Undo** command to backtrack through your actions



### Free Play



*Presenter: Once all participants have launched Sketchpad, allow about 10 minutes for Free Play. Stop once all the participants have completed this section.*

To begin learning about Sketchpad, you'll want to experiment with the various tools. We'll start this tour by encouraging you to do just that, giving you just a little guidance.

1. Start Sketchpad. If Sketchpad is already running, choose **New Sketch** from the File menu.
2. Use the **Point**, **Compass**, and **Straightedge** tools to draw objects in the sketch plane. (Make sure you use all three straightedge tools—the **Segment**, **Ray**, and **Line**.) After you've used each tool at least once, use the **Selection Arrow** tool to drag different parts of your sketch. Then try the following tasks:
  - What happens when you use the **Selection Arrow** to click on the intersection of two straight objects, two circles, or a straight object and a circle?
  - What happens when you hold down the Shift key when constructing a segment or other straight object? Is a straight object constructed this way restricted when dragged in the future, or can it be dragged like any other straight object?
  - Use the **Compass** tool to draw a circle. Use the **Point** tool to construct a point on the circumference of that circle. Use the **Selection Arrow** to drag, in turn, each of the three points. How does dragging each point affect the construction?
  - Draw a segment with one endpoint "on top" of another segment or circle. How does this endpoint move when you drag it with the **Selection Arrow**? What happens if you move the segment or circle on which it's constructed?

You'll learn more about each of these tools throughout the tours.

- What happens when you use the **Selection Arrow** to click on an object? What happens if you click again on the same object? What happens if you click in empty space?
- Using the **Point** tool, construct three independent points. Use the **Selection Arrow** to select and drag just one of those points. Now drag two of them at the same time. Then try dragging all three. Now what if you only want to drag one?
- Using the **Arrow** tool again, select one or a small number of objects. See what commands are available in the Construct menu and try some out.

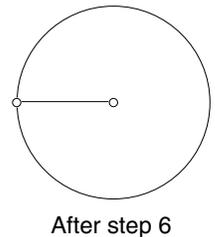


*Presenter: Stop here and answer questions. Go over those items above that seemed to generate the most questions. Make sure participants understand how selection highlighting determines which objects are dragged and how target highlighting determines which object gets constructed on which other object. Mention that selection also controls the availability of menu commands. Allow 15 minutes for the rest of this activity.*

### Getting Serious

3. When you're ready to move on, do the following: Choose **Undo** from the Edit menu—or, better yet, press Ctrl+Z on Windows or ⌘+Z on Mac—to backtrack through your work.
4. You should now be working in a blank sketch. Choose the **Segment** tool from the Toolbox (make sure it's the **Segment** tool, not the **Ray** or **Line** tool).
5. Draw a segment to form the bottom side of your square.
6. Using the **Compass** tool, click first on the right endpoint and then on the left endpoint of the segment. Each endpoint should highlight when you click on it. This constructs a circle centered at the right endpoint of the segment and passing through the other endpoint.
7. Using the **Selection Arrow** tool, drag each of the two points in the sketch to be certain that the circle is attached to the segment. If it's not, and there are actually more than two points in the sketch, **Undo** the last step in your sketch and redo step 6.
8. Click with the **Arrow** in blank space to deselect all objects.

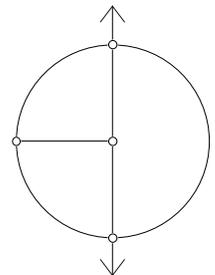
  
After step 5



After step 6

### Constructing Perpendicular and Parallel Lines

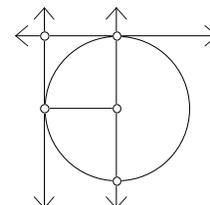
Next you'll construct a line through the center of the circle and perpendicular to the segment. Notice that we had to state both of those pieces of information—"through the center of the circle" and "perpendicular to the segment"—to be clear which line we meant. Similarly, Sketchpad needs two items to be selected—a point and a straight object—in order to construct a perpendicular (or parallel) line.



After step 10

To construct a parallel line, Sketchpad needs to be given a straight object to be parallel to and a point to pass through (just as with perpendicular lines).

9. Select the segment and its right endpoint by clicking on them with the **Arrow**. Choose **Perpendicular Line** from the Construct menu.
10. With the line still selected, select the circle as well. Then choose **Intersections** from the Construct menu.
11. Deselect all objects by clicking in blank space. Select the topmost point just created and the original segment. Then choose **Parallel Line** from the Construct menu.
12. Use either **Perpendicular Line** or **Parallel Line** to construct a vertical line through the left endpoint of the original segment.
13. Deselect all objects. Select the two lines from the last two steps and choose **Intersection** from the Construct menu.



After step 13

### Completing the Construction and the “Drag Test”

Before you finish your square, let’s see if it passes the “drag test.” As you did in step 7, use the **Selection Arrow** to drag each of the four vertices. Does everything stick together? If so, great! If not, **Undo** steps until you figure out what went wrong and try constructing the figure again. Then move ahead to finish the figure.

14. Select everything *except* the four corners of the square and the original segment.
15. Choose **Hide Objects** from the Display menu. You should see only what’s in the figure at right. If there’s more in your sketch, hide whatever is extra. If there’s less, choose **Undo Hide Objects** from the Edit menu and try step 14 again.
16. Using the **Segment** tool, construct the remaining three sides of the square.



After step 15

Voilà! You have now made your first Sketchpad construction!

Is your figure a square no matter how you drag its parts? If not, it’s not a true Sketchpad square. This is the essence of the “drag test.”

17. Use the **Selection Arrow** tool to once again drag different parts of your square.
18. Choose **Select All** from the Edit menu to select the entire square. Now choose **Animate Objects** from the Display menu. Animating a construction in this way is an alternative to the drag test.
19. Choose **Stop Animation** from the Display menu. (You could also click on the Stop button in the Motion Controller.)
20. Keep your sketch open, as you may use it in a later activity.



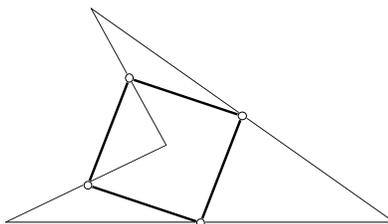
*Presenter: Stop here, review the construction of the square, and answer questions. Review the importance of selecting the appropriate objects (and only the appropriate objects) before choosing a menu command. Give an example of a square that doesn’t pass the drag test.*

### Further Challenges

- See if you can construct the following objects: a rectangle, a parallelogram, a rhombus, a trapezoid, and an equilateral triangle.

## Tour 2: A Theorem About Quadrilaterals

Sometimes students will have trouble remembering or understanding a particular theorem. Discovering the theorem themselves or actively exploring its consequences can make a huge difference in their level of recall and understanding. This tour challenges you to discover a theorem. As you solve the challenge, you'll learn many new Sketchpad features.



When the midpoints of the sides of a quadrilateral are connected, the resulting shape is always a \_\_\_\_\_.

### What You Will Learn

- How to construct a polygon using the **Segment** tool
- How to show an object's label
- How to measure lengths and angles
- How to create a caption
- How to apply formatting options to text

The theorem stated above has a key word left blank. Our goal is to use Sketchpad to discover what word fills the blank. Don't worry if this theorem is unfamiliar or, on the other hand, if it's too simple. The point is to learn about how to use Sketchpad for making and testing conjectures.

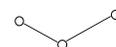
### Constructing a General Quadrilateral



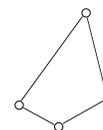
*Presenter: Allow 10 minutes for the first three sections.*

A "general quadrilateral" is a quadrilateral with no special properties, such as congruent sides or right angles.

1. Start Sketchpad if it isn't already running. If it is running, choose **New Sketch** from the File menu.
2. Use the **Segment** tool to draw a segment.
3. Construct a second segment that shares one endpoint with the first.
4. Construct two more segments to complete the quadrilateral.
5. Using the **Arrow** tool, drag some of the points and segments to make sure the figure holds together—does it pass the drag test?



After step 3



After step 5

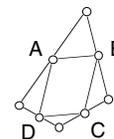
### Constructing the Midpoint Quadrilateral

Now that you have a quadrilateral, recall the theorem you're exploring. It begins "When the *midpoints* of the sides of a quadrilateral are *connected* . . ." You will continue by constructing the four midpoints and then connecting them.

6. Deselect all objects by clicking in blank space. Select the four sides of the quadrilateral in clockwise or counter-clockwise order.

You can also use the **Text** tool to show the labels. Click on an object with the **Text** tool to display (or later hide) its label. Double-click the label itself to change the label.

7. Choose **Midpoints** from the Construct menu.
8. With the four midpoints still selected, choose **Show Labels** from the Display menu.
9. With the four midpoints *still* selected, choose **Segments** from the Construct menu.
10. Use the **Arrow** tool to drag each point in your figure. Especially make sure that the midpoints are really midpoints. (Can you drag them so that they are *not*?)



After step 9

### Dragging and Measuring to Confirm Your Conjecture

You may already have a hypothesis about what kind of quadrilateral the inner figure is. Be careful, though, because it may be that shape only for a particular shape of the outer quadrilateral. You need to convince yourself that your conjecture holds *no matter what shape the outer quadrilateral is*. This requires dragging. To gain even more confidence in your answer, you'll take measurements as well.

11. Drag various parts of the original quadrilateral around, keeping an eye on the inscribed figure.
12. Now, select the four inner segments and choose **Length** from the Measure menu.  $m \overline{AB} = 1.15 \text{ cm}$
13. Deselect all objects. Then select three consecutive midpoints and choose **Angle** from the Measure menu.  $m \angle ABC = 84.61^\circ$
14. Select three consecutive midpoints again, with a different midpoint as the second selected point (vertex), and choose **Angle**. Repeat this step two more times, making sure a different midpoint is the second point selected each time.
15. Choose **Calculate** from the Measure menu.  $m \angle ABC + m \angle BCD = 180.00^\circ$   
Click on the measure for  $\angle ABC$ , then click on the + sign (or type it on the keyboard), then click on the measure for  $\angle BCD$ , then click OK. What does this tell us?
16. Drag various parts of the original quadrilateral, making it wide, skinny, concave, and so on. What happens to the measurements? What changes, and what stays the same? Are you more confident of your conjecture now? As confident as you may feel, do you think you've *proven* the theorem?

When selecting three points to measure an angle in Sketchpad, the second selected point is the vertex.



*Presenter: Stop here, answer questions, and ask different participants to state their observations or conclusions. Don't let them jump to the statement of the theorem until they've stated specific observations. Allow 10 minutes for the next section.*

### Creating a Caption and Using the Text Palette

To finish things off, let's create a caption for this sketch so that others who see it will know what's going on.



17. Choose the **Text** tool from the Toolbox.
18. Double-click in a blank part of the sketch plane to start a new caption.
19. Use your keyboard to type the caption, perhaps stating your conjecture about the inner quadrilateral. Click outside the caption to close it.

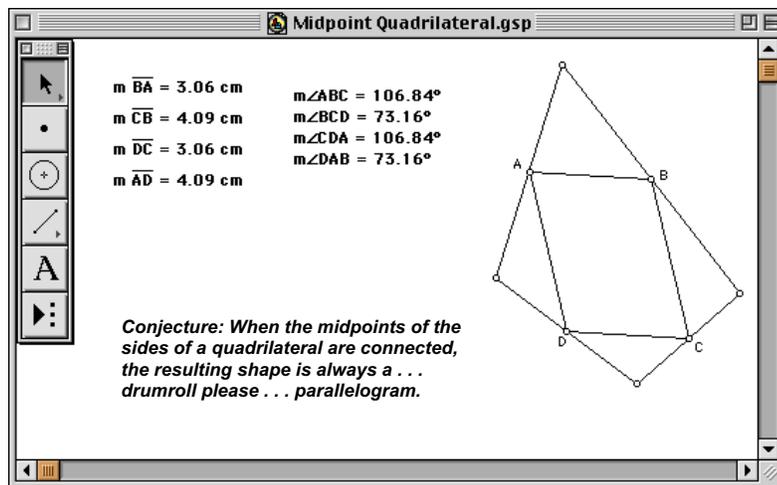
You may have noticed a new set of tools when working with captions (or other text objects). This is called the *Text Palette*. You can use the Text Palette to apply formatting options—such as italics, font size, font color, and more advanced mathematical formatting—to text. (You can manually show or hide the Text Palette using **Show Text Palette** or **Hide Text Palette** from the Display menu.)



Text Palette (Windows)

While you're exploring the Text Palette, you may also want to experiment with math notation. Click on the button on the far right of the Text Palette to show the math notation tools.

20. Use the Text Palette to change the font, text color, and other available settings of the caption. Changes will affect the entire caption because it's currently highlighted. Using the **Text** tool, click and drag within the caption to highlight a range of text and apply changes just to that text.
21. When you're done, leave your sketch open in case you want to refer to it later.



*Presenter: Stop and answer question. Emphasize the difference between captions and labels, and ask participants to describe possible uses they can think of for using captions in sketches. You may want to demonstrate some of the features of the Text Palette.*

### Further Challenges

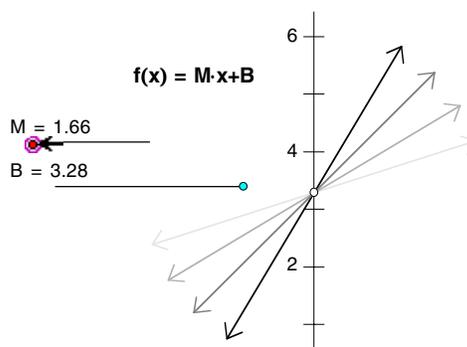
- In the sketch you just completed, make each segment and the measurement of its length the same color so that they're easy to match up visually.
- Construct a triangle and measure its angles. Show that the sum of the angles is  $180^\circ$ .
- Construct a square. Use the Calculator to show that the ratio of the length of the diagonal to the length of a side is  $1.4142\dots$
- Construct a circle and a radius of that circle. Label the center and the radius. Measure the circumference and the radius and use the Calculator to find  $c/2r$ .
- Can you *prove* your conjecture? Add some things to your sketch to make your proof more intuitively obvious.

## Tour 3: Algebra Potpourri

Don't let the name of the program—The Geometer's Sketchpad—fool you. Sketchpad has a host of tools for exploring algebra, trigonometry, and calculus, both symbolically (with equations) and graphically. In this tour, you'll sample several of Sketchpad's algebra features.

### What You Will Learn

- How to create an  $x$ - $y$  coordinate system and measure the coordinates of a point
- How to define and plot functions
- How to plot two measurements as an  $(x, y)$  point in the plane
- How to construct a locus



### A Simple Plot in the Coordinate ( $x$ - $y$ ) Plane



*Presenter: Allow 10 minutes for the first two sections.*

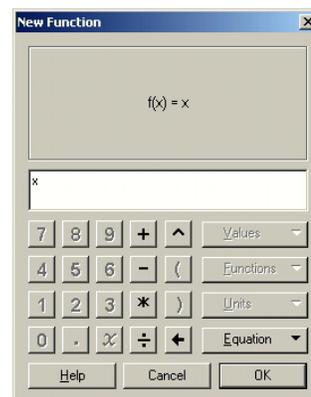
At the heart of “visual” algebra is the  $x$ - $y$  (or *Cartesian*, or *coordinate*) plane. In this section, you'll define a coordinate plane, measure a point's coordinates on it, and plot a simple function.

1. Open a new sketch and choose **Define Coordinate System** from the Graph menu.
2. Using the **Point** tool, create a point somewhere other than on an axis.
3. With the point still selected, choose **Coordinates** from the Measure menu. Drag the point to see how the coordinates change.

Actually, you can plot a function without first creating a coordinate system. Sketchpad creates coordinate systems automatically whenever needed.

Now that there's a coordinate system, let's plot a simple equation on it:  $y = x$ .

4. Choose **Plot New Function** from the Graph menu.
5. Click on  $x$  in the Function Calculator (or type  $x$  from your keyboard). Click OK.
6. Select the plot you just created and the independent point (whose coordinates you measured). Then choose **Merge Point To Function Plot** from the Edit menu. Drag the point along the plot and observe its coordinates.



You may notice that the plot's domain is restricted. To change the domain, drag the arrows at either end of the plot.

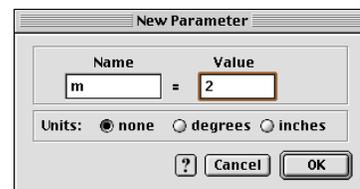
7. Drag the unit point—the point at  $(1, 0)$ —to the right and left and observe how the scale of the coordinate system changes.
8. Release the unit point when the  $x$ -axis goes from about  $-10$  to  $10$ .

## Plotting a Family of Curves with Parameters

Plotting one particular equation,  $y = x$ , is all well and good. But the real power of Sketchpad comes when you plot *families* of equations, such as the family of lines of the form  $y = mx + b$ . You'll start by defining parameters  $m$  and  $b$  and editing the existing function equation to include the new parameters. Then you'll *animate* the parameters to see a dynamic representation of this family of lines.

A parameter is a type of variable that takes on a fixed value.

9. Choose **New Parameter** from the Graph menu. Enter  $m$  for Name and  $2$  for Value and click OK.



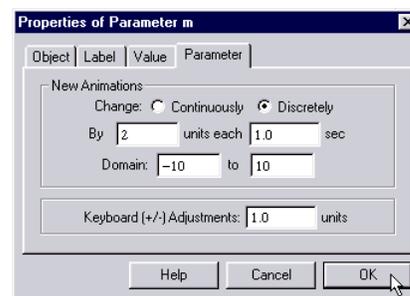
The New Parameter dialog box (Mac)

**Edit Function** appears as **Edit Calculation**, **Edit Parameter**, or **Edit Plotted Point** when one of those types of objects is selected.

10. Use the same technique to create a parameter  $b$  with the value  $-1$ .
11. Select the function equation  $f(x) = x$  (select the equation itself, not its plot) and choose **Edit Function** from the Edit menu.
12. Edit the function to be  $f(x) = m \cdot x + b$ . (Click on the parameters  $m$  and  $b$  in the sketch to enter them. Use  $*$  from the Function Calculator or keyboard [Shift+8] for multiplication.) Click OK.
13. Change  $m$  and  $b$  (by double-clicking them) to explore several different graphs in the form  $y = mx + b$ , such as  $y = 5x + 2$ ,  $y = -1x - 7$ , and  $y = 0.5x$ .

You can learn a lot by changing the parameters manually, as you did in the previous step. But it can be especially revealing to watch the plot as its parameters change smoothly or in steps.

14. Deselect all objects. Then select the parameter equation for  $m$  and choose **Animate Parameter** from the Display menu. What happens?
15. Press the Stop button to stop the animation.
16. Select  $m$  and choose **Properties** from the Edit menu. Go to the Parameter panel and change the settings so they resemble those at right. Click OK.
17. Again choose **Animate Parameter**.
18. Continue experimenting with parameter animation. You might try, among other things, animating both parameters simultaneously or tracing the line as it moves in the plane.



The Parameter Properties dialog box (Windows)



*Presenter: Stop here and answer questions. You may wish to demonstrate modeling  $y = mx + b$  using sliders instead of parameters (perhaps using sliders from the sample sketch **Sliders.gsp**). Allow about 15 minutes for the remainder of the activity.*

## Functions in a Circle

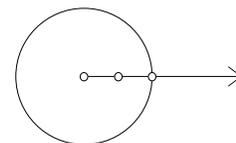
How does the radius of a circle relate to its circumference? To its area? These are examples of geometric relationships that can also be thought of as functions and studied algebraically.

You'll start by constructing a circle whose radius adjusts continuously along a straight path.

19. In a new sketch, use the **Ray** tool to construct a horizontal ray.
20. With the ray still selected, choose **Point On Ray** from the Construct menu.
21. With the **Arrow** tool, click in blank space to deselect all objects. Then select, in order, the ray's endpoint and the point constructed in the previous step. Choose **Circle By Center+Point** from the Construct menu.

In each case, use the **Arrow** tool to select the circle (and nothing else), then choose the appropriate command in the Measure menu.

22. Measure the circle's radius, circumference, and area.
23. Drag the circle's radius point and watch the measurements change.



After step 21

Next we'll explore these measurements by plotting.

24. Select, in order, the radius measurement and the circumference measurement. Choose **Plot As (x, y)** from the Graph menu. Can you see your point? If not, it might be off the screen.

In a rectangular grid, the x- and y-axes can be scaled independently.

25. Choose **Rectangular Grid** from the Graph | Grid Form submenu. Drag the new unit point at (0, 1) down until you can see the plotted point from the previous step.

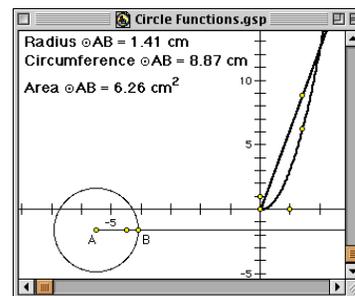
Choose **Erase Traces** from the Display menu to erase this trace at any time.

26. Select the plotted point and choose **Trace Plotted Point** from the Display menu. Drag the radius point and observe the trace.

In situations such as this (where you trace something as a point moves along a path) you can often get a smoother picture by creating a *locus*.

A Sketchpad locus is a *sample* of possible locations of the selected object. To change the number of samples plotted, select the locus, choose **Properties** from the Edit menu, and go to the Plot panel.

27. Select the plotted point and the radius point; then choose **Locus** from the Construct menu. What is the slope of the line containing this locus?
28. Repeat steps 24 and 27, except this time explore the relationship between the radius and area measurements. How do the two curves compare? For what radius do the numerical values of a circle's circumference and area equal each other (ignoring units)?



## Further Challenges

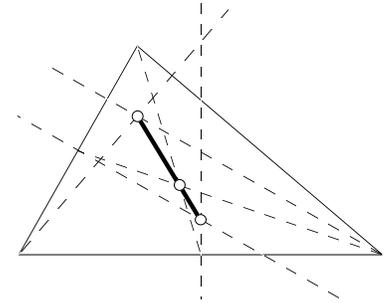
- Graph pairs of parallel lines and show that their slopes are the same.
- Graph pairs of perpendicular lines and show that their slopes are negative reciprocals.
- Graph a parabola of the form  $y = a(x - h)^2 + k$ , using parameters for  $a$ ,  $h$ , and  $k$ .

## Tour 4: Investigating Triangle Centers

In this tour, you'll continue honing your skills at geometric construction by constructing a triangle's centroid and other classical triangle centers. Then you'll harness Sketchpad's power in two important new ways: first, by turning your constructions into *custom tools*, then by making a *multi-page document*.

### What You Will Learn

- How to change Sketchpad's Preferences
- How to construct a triangle's centroid and at least one other classical triangle center
- How to turn a construction into a custom tool and how to use that tool to quickly reproduce the construction
- How to show the Script View of a custom tool and how to step through a script
- How to create a multi-page Sketchpad document



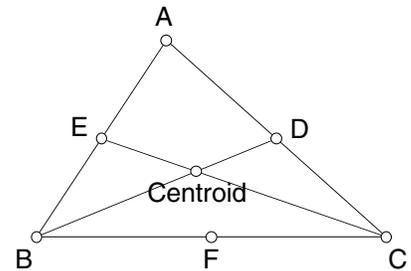
### Constructing a Centroid



*Presenter: Allow 15 minutes for the first two sections.*

1. Set up Sketchpad so that points are automatically labeled when constructed. To do this, choose **Preferences** from the Edit menu, go to the Text panel, and check For All New Points. Before clicking OK, you may want to explore what other options are available in Preferences.
2. Use the **Segment** tool to construct a triangle.
3. Construct the midpoints of the three sides.
4. Use the **Segment** tool to construct just two of the triangle's three medians. (A median connects a vertex with the midpoint on the opposite side.)
5. Using the **Point** tool or the **Selection Arrow** tool, click at the intersection of the two medians. This constructs the point of intersection.
6. Label the intersection point "Centroid." Before clicking OK, check Use Label In Custom Tools. (You'll see why soon.)
7. Construct the third median.

A shortcut: With the **Segment** tool active, choose **Select All Segments** from the Edit menu. Then choose **Midpoints** from the Construct menu; all three midpoints are constructed at once.



You can also select the entire figure and choose **Animate Objects** from the Display menu.

- You're ready to make a conjecture: The three medians in a triangle intersect in a single point. But is that true in all triangles?
8. Use the Drag Test to test your conjecture.



## Turning It into a Custom Tool

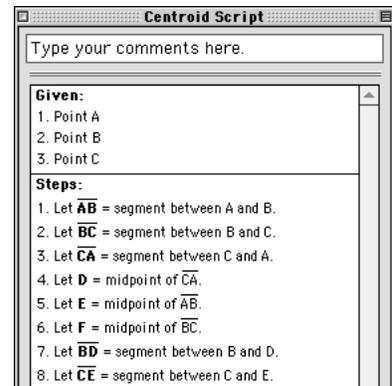
It doesn't take *too* long to construct a centroid, but if you needed to construct a lot of them for different triangles, it would be nice to do so quickly. *Custom tools* allow you to do just that, as you'll see in the next section.

9. Select the entire figure. (Either choose **Select All** from the Edit menu or use a Selection Rectangle.)
10. Choose **Create New Tool** from the Custom Tools menu (the bottom tool in the Toolbox). Name the tool **Centroid** and click OK.



11. Click on the **Custom** tools icon to choose your new tool. Now click three times in the sketch plane. You should get a construction just like the original one.
12. Use the tool a few more times, clicking sometimes in blank space and other times on existing points or segments. Choose **Undo Centroid** and then **Redo Centroid** from the Edit menu and notice that these undo and redo the entire construction at once. Notice also that while most of the points are given new labels, the centroid is always labeled *Centroid*. This is because of what you did in step 6.

13. Choose the **Selection Arrow** tool and click in blank space to deselect all objects. Then choose **Show Script View** from the Custom Tools menu. You'll see a window called Centroid Script. This is a readable description of the construction performed by the custom tool **Centroid**. Take a moment to look through the script and see how it relates to the construction.



14. Move the Script View window off to the side (by dragging it by its title bar) but don't close it. Select three points in the sketch. All three Givens should be highlighted in the Script View window and you should see two new buttons at the bottom of the window.
15. Click on the Next Step button as many times as it takes to get through the entire script. "Stepping" through a script is a great way to understand how an unfamiliar tool works or to check student work.



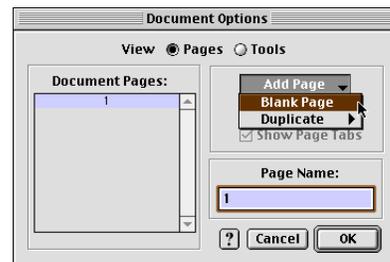
*Presenter: Stop here and answer questions. You may wish to demonstrate what would have happened if participants had constructed all three medians before constructing the intersection point in step 5. (Namely, if you click with the **Arrow** tool, you can't tell to which two segments the intersection belongs; with the **Point** tool—or selecting two segments and using the **Intersection** command—you can). You may also wish to briefly introduce the concepts of givens, intermediate objects, and results and perhaps demonstrate what happens when you create a centroid tool with fewer objects selected (just the vertices, sides, and centroid).*

*Allow as much time as you can for the next section. The more time, the more triangle centers/custom tools each participant will be able to make.*

## Creating a Multi-Page Document

When you visit a web site, it's rare to see all the content on one page—it's usually distributed over many linked pages. Similarly, in Sketchpad, you can create *multi-page documents* with several pages of related content. Students can use multi-page documents to create portfolios of their Sketchpad work.

16. Clean up your sketch a bit, perhaps by **Undoing** back to the point at which there was just one triangle with its medians and centroid.
17. Choose **Document Options** from the File menu. In the dialog box that appears, choose **Blank Page** from the Add Page pop-up menu, then click OK. You'll now be on page 2 of a two-page Sketchpad document.
18. Click on the tab marked "1" at the bottom of the window to go back to the previous page, then return to the new, blank page.
19. Choose one of the three remaining classical triangle centers listed below. Construct it on page 2 of your document using the Toolbox tools and Construct menu commands. Then turn your construction into a new custom tool, just as you did with the centroid.



*Circumcenter:* The point of concurrency (intersection) of the perpendicular bisectors of the three sides of a triangle.

*Incenter:* The point of concurrency of the three angle bisectors in a triangle.

*Orthocenter:* The point of concurrency of the three altitudes in a triangle.



*Presenter: Stop here and answer questions. You may wish to demonstrate how to use Document Options to add a duplicate page and to rename and reorder pages.*

## Further Challenges

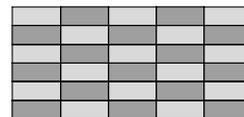
- The centroid divides each median into two smaller subsegments. Use measurements to explore the length ratio of any pair of subsegments.
- The three medians of a triangle divide it into six smaller triangles, each defined by one vertex of the original triangle, one midpoint of an original side, and the centroid. Construct these six interiors, then measure their areas. What do you notice? Why is this true?
- If you constructed a circumcenter in step 19, use it to construct the *circumcircle* of its triangle—the unique circle that perfectly circumscribes the triangle.
- If you constructed an incenter in step 19, use it to construct the *incircle* of its triangle—the unique circle that perfectly inscribes the triangle.
- If you constructed an orthocenter tool in step 19, use it to explore the following question: Where is the orthocenter of any triangle defined by two of the vertices and the orthocenter of another triangle? Why?
- Continue adding pages and tools to your document until there are four pages and four tools—one for every classical triangle center.
- Once you've completed the previous challenge, apply each of the four tools to a single triangle. Three of the four centers will lie along the same segment, called the *Euler segment*. Which three centers are these? Use measurements to explore the distance relationships between these three centers.

## Tour 5: Introducing Transformations

Up until now, we've focused on the compass-and-straightedge-derived commands in the Construct menu for doing our geometry. In this tour, you'll explore the Transform menu commands, based in *transformational geometry*, and ultimately tile the plane with parallelograms.

### What You Will Learn

- How to reflect figures across marked mirrors
- How to translate figures by marked vectors
- How to use the **Merge** command to alter the geometric properties of figures in a sketch



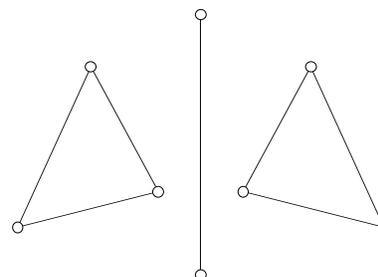
### Reflections of a Triangle

In which you use Sketchpad to explore the other side of the looking glass . . .



*Presenter: Allow 20 minutes for the first two sections.*

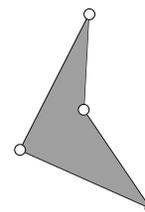
1. In a new sketch, use the **Segment** tool to draw a triangle on the left side of the sketch window.
2. Draw a vertical segment in the middle of your screen.
3. While this segment is still selected, choose **Mark Mirror** from the Transform menu. A brief animation indicates that the segment has been *marked* as a mirror.
4. Select the whole triangle using a Selection Rectangle.
5. With your triangle selected, choose **Reflect** from the Transform menu. A mirror image of your triangle should appear on the other side of your "mirror" segment.



6. Alter your original triangle. What happens to the mirror image? Move your triangle around, make it bigger, smaller, move it closer to the mirror, further away, across. Try to predict the associated movements of the image triangle. What happens when you drag the image triangle? The mirror segment?
7. Draw a horizontal mirror and reflect both triangles across it. Explore and predict the motions of the three image triangles when you change your original triangle. Explore what happens when you change the relative positions of the two mirror segments and make them nonperpendicular.

### Dynamic Translations

8. Add a new, blank page to your document.
9. Construct a general quadrilateral interior. To do this, choose the **Point** tool, then hold down the Shift key as you click in four different spots in the sketch window. With the four points selected, choose **Quadrilateral Interior** from the Construct menu.



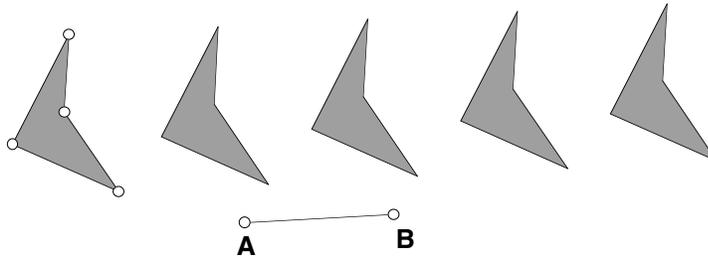
Holding down the Shift key while you draw helps keep the segment vertical (though it can still be dragged any which way later on).

Students enjoy drawing stick figures with the **Segment** and **Compass** tools, reflecting them across a mirror line, then dragging. You might enjoy this yourself!

Holding down the Shift key while creating objects with Toolbox tools keeps previously selected objects selected.

A shortcut: Select the interior and choose **Select Parents** from the Edit menu. Now choose **Segments** from the Construct menu.

10. Construct the four sides of your polygon.
11. Use the **Segment** tool to draw a segment in a blank area of the sketch plane. Show the labels of the two endpoints of the new segment, then select them. (Note the order of selection.)
12. Choose **Mark Vector** from the Transform menu. A brief animation from the first selected point to the second selected point indicates that the vector has been marked.
13. Select the sides and interior of your polygon. Choose **Translate** from the Transform menu. The Translate dialog box appears. Click Translate to translate by the marked vector.
14. Drag an endpoint of the segment defining the translation vector. What happens to the translated image?
15. Select the translated sides and interior and translate them again by the marked vector. (You won't need to mark the vector again—just go straight to the **Translate** command.) Do this several times in a row to get several (five to eight) translated images.



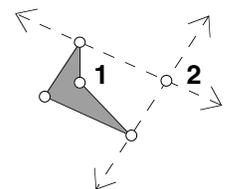
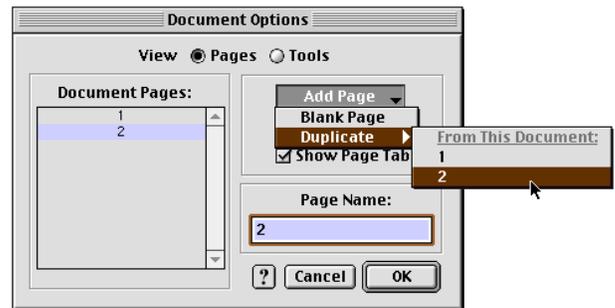
*Presenter: Stop here and answer questions. Allow 20 minutes for the next section. It's a very tricky section, so be ready to give extra support!*

## Tiling Parallelograms

Everyone's human. What if you got to this point and realized, human that you are, that you goofed, that you meant to make the original polygon a parallelogram? Would you have to start over? Fortunately, thanks to the **Merge** command, you wouldn't.

This is a good thing to do when you think a sketch may branch out in different directions. You can even "import" a page from one document into another.

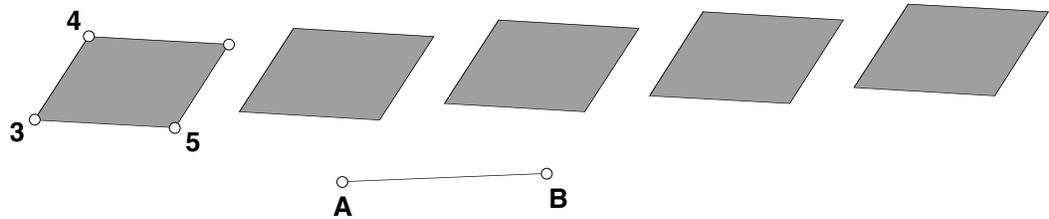
16. Add a duplicate of the current page to your document.
17. Drag point *B* so that the images are quite spread apart.
18. To turn your quadrilateral into a parallelogram, you'll only have to "redefine" one of its vertices—the other three will remain unchanged. Pick which three should remain as they are, then *construct* a new point to be the fourth vertex of the parallelogram. (See the figure at right for one approach involving two parallel lines.) Hide everything new in this step but the point just constructed.



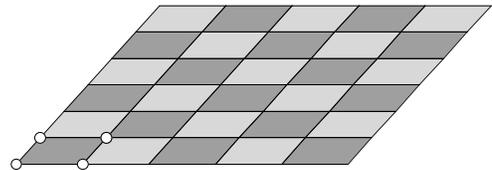
19. Select the “other” quadrilateral vertex and the newly constructed point (points 1 and 2 in the figure at step 18) and choose **Merge Points** from the Edit menu. Use the Drag Test to confirm that your polygons are indeed parallelograms no matter what.

Parallelograms are special in that they tile the plane. In other words, they can be arranged in such a way that they fill the entire plane without overlaps or holes. You’ll demonstrate this fact in the next section.

20. Drag objects until your sketch looks more or less like the figure below.



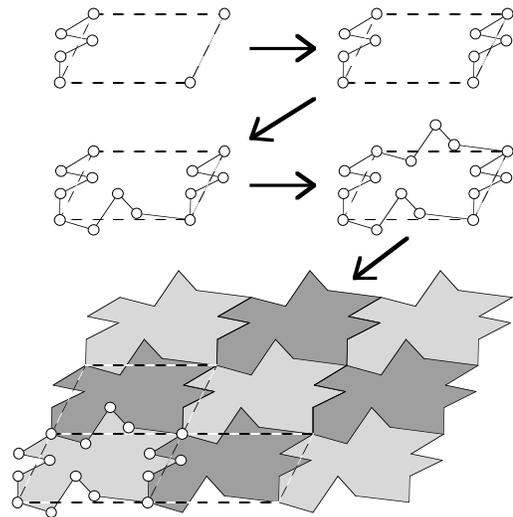
21. Mark a new vector from point 3 to point 4 (see the figure above).
22. Translate everything in the sketch (except  $\overline{AB}$  and its endpoints) by the marked vector. Repeat several (five to eight) more times.
23. Drag point  $B$  until your parallelograms indeed tile the plane. How many such locations are there for point  $B$ ?
24. To fix your construction so that the parallelograms always tile the plane, no matter what is dragged, merge point  $A$  to point 3, then merge point  $B$  to point 5.



*Presenter: Stop here and answer questions.*

### Further Challenges

- Do all quadrilaterals tile the plane or just certain special ones? Use Sketchpad to explore this question.
- Shapes that tile the plane can form the basis of what might be called an *Escher-like* tessellation. Study the figure at right, then create your own Escher-like tessellation.



*Note: This activity was adapted from a series of activities designed by John Olive, Ph.D., University of Georgia, Athens, Georgia, and was first used at a workshop conducted at the International Conference on Technology in Mathematics Teaching at the University of Birmingham, England.*

## Tour 6: A Sine Wave Tracer

One way to use Sketchpad in your courses is to have students work through activities from one of the curriculum modules (see the inside back cover of this booklet for the complete list). This tour introduces you to Sketchpad animation as you work through an activity from the book *Exploring Geometry with The Geometer's Sketchpad*. The activity is reproduced verbatim, starting with "In this exploration . . .," followed by a special extension section.

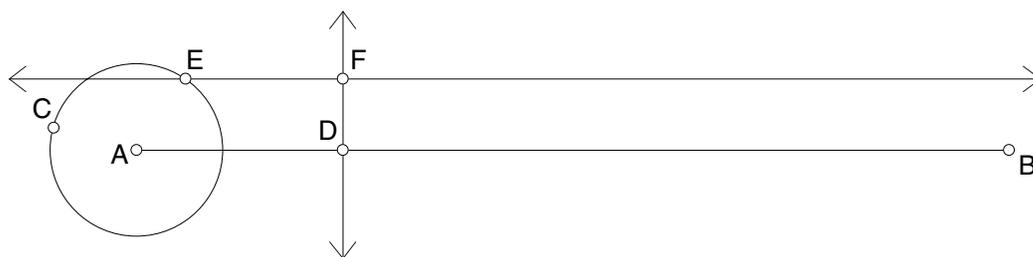


*Presenter: Allow 20 minutes for the first section. Make sure to tell participants what they should do with the questions (Q1, Q2, and so on). Mention that in the actual activity, there is space after every question for written responses.*

In this exploration, you'll construct an animation "engine" that traces out a special curve called a *sine wave*. Variations of sine curves are the graphs of functions called *periodic functions*, functions that repeat themselves. The motion of a pendulum and ocean tides are examples of periodic functions.

### Sketch and Investigate

1. Construct a horizontal segment  $AB$ .



2. Construct a circle with center  $A$  and radius endpoint  $C$ .
3. Construct point  $D$  on  $\overline{AB}$ .
4. Construct a line perpendicular to  $\overline{AB}$  through point  $D$ .
5. Construct point  $E$  on the circle.
6. Construct a line parallel to  $\overline{AB}$  through point  $E$ .
7. Construct point  $F$ , the point of intersection of the vertical line through point  $D$  and the horizontal line through point  $E$ .

Select point  $D$  and  $\overline{AB}$ ; then, in the Construct menu, choose **Perpendicular Line**.



- **Q1** Drag point  $D$  and describe what happens to point  $F$ .

- Q2** Drag point  $E$  around the circle and describe what point  $F$  does.

- Q3** In a minute, you'll create an animation in your sketch that combines these two motions. But first try to guess what the path of point  $F$  will be when point  $D$  moves to the right along the segment at the same time that point  $E$  is moving around the circle. Sketch the path you imagine below.

Don't worry, this isn't a trick question!

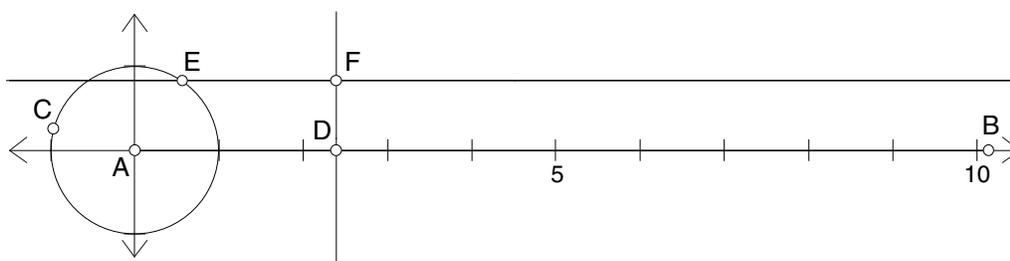


Select points  $D$  and  $E$  and choose **Edit | Action Buttons | Animation**. Choose **forward** in the Direction pop-up menu for point  $D$ .

8. Make an action button that animates point  $D$  forward along  $\overline{AB}$  and point  $E$  forward around the circle.
9. Move point  $D$  so that it's just to the right of the circle.
10. Select point  $F$ ; then, in the Display menu, choose **Trace Intersection**.
11. Press the Animation button.

**Q4** In the space below, sketch the path traced by point  $F$ . Does the actual path resemble your guess in Q3? How is it different?

12. Select the circle; then, in the Graph menu, choose **Define Unit Circle**. You should get a graph with the origin at point  $A$ . Point  $B$  should lie on the  $x$ -axis. The  $y$ -coordinate of point  $F$  above  $\overline{AB}$  is the value of the sine of  $\angle EAD$ .



**Q5** If the circle has a radius of 1 grid unit, what is its circumference in grid units? (Calculate this yourself; don't use Sketchpad to measure it because Sketchpad will measure in inches or centimeters, not grid units.)

13. Measure the coordinates of point  $B$ .
14. Adjust the segment and the circle until you can make the curve trace back on itself instead of drawing a new curve every time. (Keep point  $B$  on the  $x$ -axis.)

**Q6** What's the relationship between the  $x$ -coordinate of point  $B$  and the circumference of the circle (in grid units)? Explain why you think this is so.



*Presenter: Stop here and answer questions. Discuss issues relating to using an activity such as this in the classroom (for example, number of students per computer, written versus oral responses to questions, and so on). Mention that most activities have an "Explore More" section at the end—this activity is an exception.*

*Allow 15 minutes for the next section.*

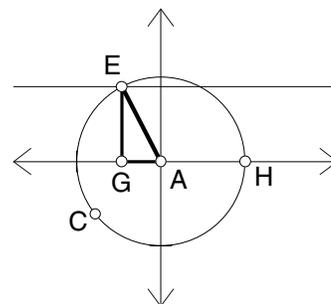
### Extension: "Special Points"

The *Exploring Geometry* activity ends there. While it wasn't designed to be a full-blown activity on unit circle trigonometry, you can nevertheless add a few things that will lead in that direction. In the next section, you'll add the right triangle associated with point  $E$  and also add a new type of button—Movement buttons—that will move point  $E$  to exact locations corresponding to angles of 30, 45, and 60 degrees.

15. Measure the coordinates of point  $E$ .
16. Construct the line through point  $E$ , perpendicular to the  $x$ -axis.

Be careful *not* to construct the intersection of the perpendicular and  $\overline{AB}$  (since  $\overline{AB}$  is on the  $x$ -axis). The easiest way to be certain is to select the perpendicular and the  $x$ -axis and choose **Intersection** from the Construct menu.

- 17. Construct the intersection point of this new line and the  $x$ -axis. Label the point  $G$ .
18. Construct the point at  $(1, 0)$  and label it  $H$ . (You can either choose **Plot Points** from the Graph menu or simply click on the spot to get the point of intersection.)
19. Measure  $\angle HAE$ .
20. Hide the perpendicular line and construct segments from  $A$  to  $E$ , from  $G$  to  $E$ , and from  $A$  to  $G$ . Make these three segments thick and red. This is your “reference triangle.”



Now you’ll create points on the circle at 30, 45, and 60 degrees. You’ll do this by rotating point  $H$  about the origin, or  $A$ .

21. Double-click point  $A$ . This marks it as a center of rotation.
22. Select point  $H$  and choose **Rotate** from the Transform menu. Type 30 and click Rotate.
23. Select point  $E$  and the new point *in that order* and choose **Movement** from the Edit | Action Buttons submenu. Leave the speed at medium. On the Label panel, change the label of the button to “30 degrees.”
24. Hide the new point (at 30 degrees), then press your button. Point  $E$  should gracefully move around the circle until  $\angle HAE$  is 30 degrees.
25. Repeat steps 22 and 23 for 45 and 60 degrees.



*Presenter: Stop here and answer questions. Discuss the differences between Animation and Movement buttons.*

### Further Challenges

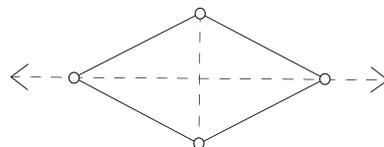
- Add to your sketch so that the animation produces a cosine curve in addition to the sine curve.
- *Trigonometry extension:* How could you add a tangent curve?

## Tour 7: Investigating Rhombus Constructions

How many different ways are there to construct a rhombus in Sketchpad? A challenge such as this can really deepen a student's understanding of a shape's definition and properties. But how can you tell how a finished construction works—how it was put together and how that relates to the properties in question? In this tour, you'll come up with several ways to construct a rhombus and learn several methods of investigating other people's constructions.

### What You Will Learn

- How to open a pre-made Sketchpad sketch
- Five techniques for investigating someone else's sketch
- What the Object Properties panel is and how to use it to navigate the family tree



### Five Investigation Techniques



*Presenter: Allow 15 minutes for the first section. One place to find **Rhombus.gsp** is in **Samples | Teaching Mathematics**. You may wish to move it for convenience.*

1. Open the sketch **Rhombus.gsp**. (Your presenter will give you more detailed instructions on where to find this file.)
2. Drag various parts of the figure. What can you deduce about the construction?

**Dragging** is the first technique for investigating a construction.

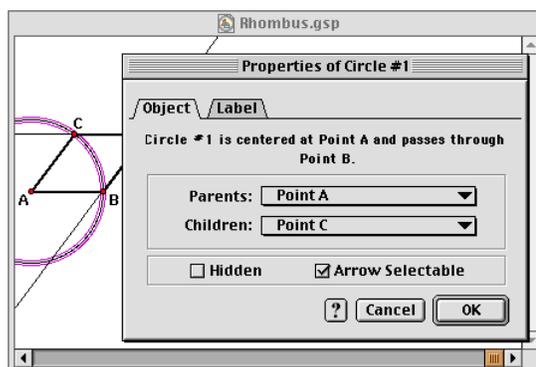
3. Choose **Show All Hidden** from the Display menu, then drag some more. What can you now deduce about the construction?

**Showing Hidden Objects** is the second technique for investigating a construction.

Another good technique for exploring constructions isn't available to you here because you opened a pre-made sketch. But when you check student work in a classroom or lab, you can use the **Undo** command in the Edit menu (**⌘+Z** Mac, **Ctrl+Z** Windows) to step back through a construction, then use the **Redo** command (**⌘+R** Mac, **Ctrl+R** Windows) to replay the student's steps.

**Undoing/Redoing** is the third technique for investigating a construction.

4. Select any *one* object in the sketch and choose **Properties** from the Edit menu. You'll see the Properties dialog box for the selected object, open to the Object panel. (Click on the Object tab if it's open to another panel.) If necessary, drag the dialog box by its title bar so that it's not obscuring the construction.



The five techniques listed in this tour aren't in a strict order. Which to use depends on the situation and your preferences.

Holding down the Shift key turns **Undo** into **Undo All** and **Redo** into **Redo All**. A good idea here is to **Undo All**, then **Redo** step-by-step.

At the top of the panel is a geometric description of the object, usually in terms of its *parents* (the objects that geometrically define the object). In the middle of the panel are two pop-up menus—the Parents pop-up menu, listing the object’s parents, and the Children pop-up menu, listing the object’s *children* (the objects geometrically defined by the object).

5. Click and hold down the mouse button on the Parents pop-up menu. Drag between the listed parents if there are more than one. Notice that the highlighted object in the menu is also highlighted (selected) in the sketch below. Release the mouse outside the pop-up menu to remain on the same object’s Object panel. (Don’t worry, though, if you end up on a different one.) Do the same thing for the Children pop-up menu.

You can also go directly to any object’s Object Properties panel by clicking on it in the sketch.

6. *Navigate the family tree.* In other words, go to a different object’s Object Properties panel by choosing it in the Parents or Children pop-up menu. Read each object’s description at the top of the panel and think about how objects relate to each other as parents and children. See if you can get to every sketch object’s Object Properties panel through these pop-up menus.

*Exploring Object Properties/Navigating the Family Tree* is the fourth technique for investigating a construction.

Recall from Tour 4, steps 14 and 15, that you can select objects in the sketch to match the tool’s givens and then walk through the script step-by-step.

7. Select the entire figure and choose **Create New Tool** from the Custom Tools menu (the bottom tool in the Toolbox). Name the tool whatever you’d like, and click on **Show Script View** before clicking OK. (If you accidentally click OK first, just choose **Show Script View** from the Custom Tools menu.) Read through the script and see how it represents the construction.

*Showing Script View* is the fifth technique for investigating a construction.



*Presenter: Stop here and answer questions. In particular, be sure to discuss the Object Properties panel and navigating the family tree. Mention that different object types have different Properties panels, and perhaps show some examples.*

*Allow as much time as you can for the next section. The more time, the more constructions/investigations each participant will be able to do.*

## Sketch and Investigate

Now it’s your turn to create new rhombus constructions and investigate the constructions of others.

To come up with new constructions, it helps to think about a rhombus’s properties, such as the relationships among its sides, angles, and diagonals, as well as any symmetries it exhibits.

8. Choose a partner.
9. *On your own*, come up with a construction for a rhombus that’s different from the one you’ve been investigating. You’ll probably want to do this in a new sketch or on a new page of **Rhombus.gsp**. When you’re done, hide everything but the sides and vertices of the rhombus.
10. Switch computers and investigate your partner’s construction. Use at least two or three of the investigation techniques listed. Your goal is not only to understand how your partner’s construction works, but also to figure out which properties of rhombuses it exploits.
11. Repeat steps 9 and 10, coming up with new rhombus constructions and trying out investigation techniques you didn’t use previously. (Note that any rhombus property can give rise to several different Sketchpad constructions.)

## What Else Can This Thing Do?

You've still seen little more than a fraction of what Sketchpad can do. In fact, we and others are finding things all the time that we'd never thought of doing with Sketchpad! At this point you've gotten your feet wet using the program. You might want to take a look at some of the sample sketches. Here's a list of some we like to show off in demonstrations, along with information you might find helpful when you're demonstrating the software.

**Samples | Sketches | Geometry | Pythagoras.gsp:** This sketch contains several unusual and fascinating proofs of the famous theorem, using dissection, shearing, and other methods.

**Samples | Sketches | Algebra | Parametric Functions.gsp:** This sketch includes two fascinating parametric functions, one on a rectangular coordinate system and one on a polar coordinate system. You can animate the parameter to trace out the function, and you can modify the function itself.

**Samples | Sketches | Trigonometry | Trigonometry Tracers.gsp:** This sketch includes constructions to trace out the sine and cosine, the tangent, and all six trig functions at once.

**Samples | Sketches | Conics | Unified Conics.gsp:** This sketch includes the locus of a parabola and an ellipse, and a construction that can be smoothly transformed between ellipse, parabola, and hyperbola.

**Samples | Sketches | Investigations | Poincare Disk.gsp:** This sketch and set of tools allows you to investigate all sorts of constructions using the Poincaré disk model of non-Euclidean geometry.

**Samples | Sketches | Advanced | Riemann Sums.gsp:** This sketch shows the upper, midpoint, and lower Riemann sums of a function on an interval. Modify the number of subintervals, the limits of the interval, or the function itself.

**Samples | Sketches | Investigations | Rainbow.gsp:** This sketch demonstrates how refraction of light within water droplets creates a rainbow.

**Samples | Sketches | Fun | Sheep Dog.gsp:** You control a sheep dog trying to herd four sheep into a pen as the sheep try to wander off whenever you're not nipping at their heels.

## Sample Activities

The following sample activities show some of the ways Sketchpad can be used to illuminate and explore topics from geometry and other areas of mathematics. These activities come from a variety of Sketchpad curriculum modules and have been chosen to show you a broad range of possible Sketchpad activities.

From *Exploring Geometry with The Geometer's Sketchpad*

**Properties of Reflection** (page 24): Measure and investigate various reflected images.

**The Euler Segment** (page 26): Which of the common circle center points are collinear, and what other relationships exist?

From *Exploring Algebra with The Geometer's Sketchpad*

**Adding Integers** (page 28): Model integers as vectors on a number line to develop an intuitive understanding.

**Points "Lining Up" in the Plane** (page 31): Observe the patterns that result when you move points so their coordinates satisfy certain conditions.

From *Pythagoras Plugged In*

**Leonardo da Vinci's Proof** (page 34): Use rotation and reflection in Leonardo's proof of the Pythagorean Theorem.

From *Exploring Conic Sections with The Geometer's Sketchpad*

**The Folded Circle Construction** (page 36): Use both physical and Sketchpad models to construct conic sections.

From *Rethinking Proof with The Geometer's Sketchpad*

**Distances in an Equilateral Triangle** (page 40): A shipwreck survivor helps you prove a theorem about triangles.

**Varignon Area** (page 43): What's the relationship between the area of an inscribed quadrilateral and the original?

From *Exploring Calculus with The Geometer's Sketchpad*

**Going Off on a Tangent** (page 46): Explore tangent lines of functions to better understand derivatives.

**Accumulating Area** (page 49): Add up the area between a function and the axis.

# Properties of Reflection

Name(s): \_\_\_\_\_

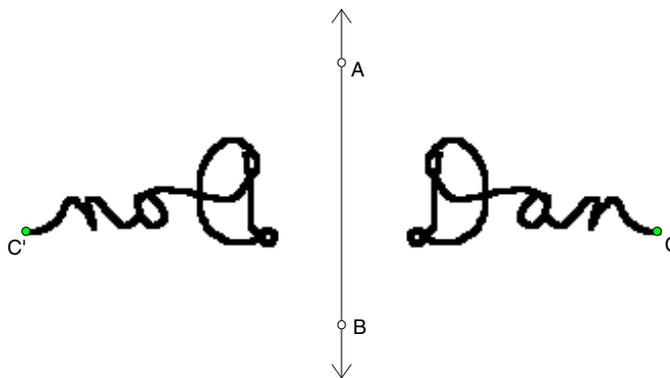
When you look at yourself in a mirror, how far away does your image in the mirror appear to be? Why is it that your reflection looks just like you, but backward? Reflections in geometry have some of the same properties of reflections you observe in a mirror. In this activity, you'll investigate the properties of reflections that make a reflection the "mirror image" of the original.

## Sketch and Investigate: Mirror Writing

1. Construct vertical line  $\overleftrightarrow{AB}$ .
2. Construct point  $C$  to the right of the line.
3. Mark  $\overleftrightarrow{AB}$  as a mirror.
4. Reflect point  $C$  to construct point  $C'$ .
5. Turn on **Trace Points** for points  $C$  and  $C'$ .
6. Drag point  $C$  so that it traces out your name.

Double-click on the line.

Select the two points; then, in the Display menu, choose **Trace Points**. A check mark indicates that the command is turned on. Choose **Erase Traces** to erase your traces.



**Q1** What does point  $C'$  trace?

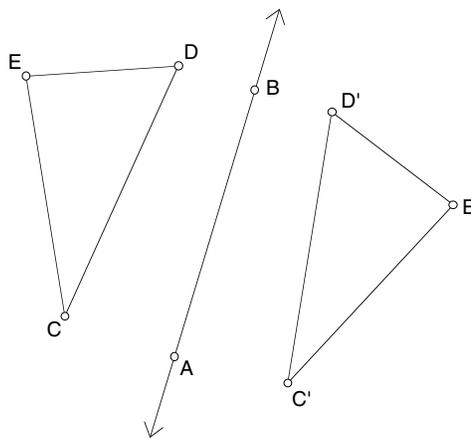
7. For a real challenge, try dragging point  $C'$  so that point  $C$  traces out your name.

## Sketch and Investigate: Reflecting Geometric Figures

8. Turn off **Trace Points** for points  $C$  and  $C'$ .
9. In the Display menu, choose **Erase Traces**.
10. Construct  $\triangle CDE$ .
11. Reflect  $\triangle CDE$  (sides and vertices) over  $\overleftrightarrow{AB}$ .
12. Drag different parts of either triangle and observe how the triangles are related. Also drag the mirror line.
13. Measure the lengths of the sides of  $\triangle CDE$  and  $\triangle C'D'E'$ .

Select points  $C$  and  $C'$ . In the Display menu, you'll see **Trace Points** checked. Choose it to uncheck it.

Select the entire figure; then, in the Transform menu, choose **Reflect**.



Select three points that name the angle, with the vertex your middle selection. Then, in the Measure menu, choose **Angle**.

→ **Q2** Measure one angle in  $\triangle CDE$  and measure the corresponding angle in  $\triangle C'D'E'$ . What effect does reflection have on lengths and angle measures?

**Q3** Are a figure and its mirror image always congruent? State your answer as a conjecture.

Your answer to Q4 demonstrates that a reflection reverses the *orientation* of a figure.

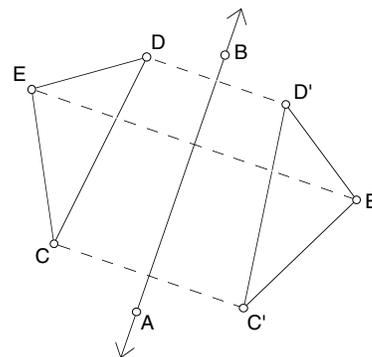
→ **Q4** Going alphabetically from  $C$  to  $D$  to  $E$  in  $\triangle CDE$ , are the vertices oriented in a clockwise or counter-clockwise direction? In what direction (clockwise or counter-clockwise) are vertices  $C'$ ,  $D'$ , and  $E'$  oriented in the reflected triangle?

**Line Weight** is in the Display menu.

→ 14. Construct segments connecting each point and its image:  $C$  to  $C'$ ,  $D$  to  $D'$ , and  $E$  to  $E'$ . Make these segments dashed.

You may wish to construct points of intersection and measure distances to look for relationships between the mirror line and the dashed segments.

→ **Q5** Drag different parts of the sketch around and observe relationships between the dashed segments and the mirror line. How is the mirror line related to a segment connecting a point and its reflected image?



### Explore More

1. Suppose Sketchpad didn't have a Transform menu. How could you construct a given point's mirror image over a given line? Try it. Start with a point and a line. Come up with a construction for the reflection of the point over the line using just the tools and the Construct menu. Describe your method.
2. Use a reflection to construct an isosceles triangle. Explain what you did.

# The Euler Segment

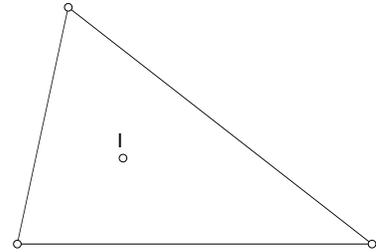
Name(s): \_\_\_\_\_

In this investigation, you'll look for a relationship among four points of concurrency: the incenter, the circumcenter, the orthocenter, and the centroid. You'll use custom tools to construct these triangle centers, either those you made in previous investigations or pre-made tools.

## Sketch and Investigate

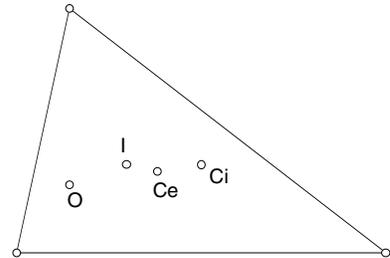
Triangle Centers.gsp can be found in Sketchpad I Samples | Custom Tools. (Sketchpad is the folder that contains the application itself.)

1. Open a sketch of yours that contains tools for the triangle centers: incenter, circumcenter, orthocenter, and centroid. Or, open **Triangle Centers.gsp**.



2. Construct a triangle.
3. Use the **Incenter** tool on the triangle's vertices to construct its incenter.
4. If necessary, give the incenter a label that identifies it, such as  $I$  for incenter.
5. You need only the triangle and the incenter for now, so hide anything extra that your custom tool may have constructed (such as angle bisectors or the incircle).

6. Use the **Circumcenter** tool on the same triangle. Hide any extras so that you have just the triangle, its incenter, and its circumcenter. If necessary, give the circumcenter a label that identifies it.



7. Use the **Orthocenter** tool on the same triangle, hide any extras, and label the orthocenter.
8. Use the **Centroid** tool on the same triangle, hide extras, and label the centroid. You should now have a triangle and the four triangle centers.

**Q1** Drag your triangle around and observe how the points behave. Three of the four points are always collinear. Which ones?

9. Construct a segment that contains the three collinear points. This is called the *Euler segment*.

**Q2** Drag the triangle again and look for interesting relationships on the Euler segment. Be sure to check special triangles, such as isosceles and right triangles. Describe any special triangles in which the triangle centers are related in interesting ways or located in interesting places.

**Q3** Which of the three points are always endpoints of the Euler segment and which point is always between them?

To measure the distance between two points, select the two points. Then, in the Measure menu, choose **Distance**. (Measuring the distance between points is an easy way to measure the length of part of a segment.)

→ **10.** Measure the distances along the two parts of the Euler segment.

**Q4** Drag the triangle and look for a relationship between these lengths. How are the lengths of the two parts of the Euler segment related? Test your conjecture using the Calculator.

### Explore More

1. Construct a circle centered at the midpoint of the Euler segment and passing through the midpoint of one of the sides of the triangle. This circle is called the *nine-point circle*. The midpoint it passes through is one of the nine points. What are the other eight? (*Hint*: Six of them have to do with the altitudes and the orthocenter.)
2. Once you've constructed the nine-point circle as described above, drag your triangle around and investigate special triangles. Describe any triangles in which some of the nine points coincide.

# Adding Integers

Name(s): \_\_\_\_\_

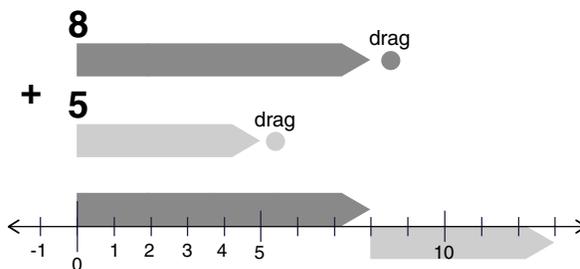
Definition: *Integers* are positive and negative whole numbers, including zero. On a number line, tick marks usually represent the integers.

They say that a picture is worth a thousand words. In the next two activities, you'll explore integer addition and subtraction using a visual Sketchpad model. Keeping this model in mind can help you visualize what these operations do and how they work.

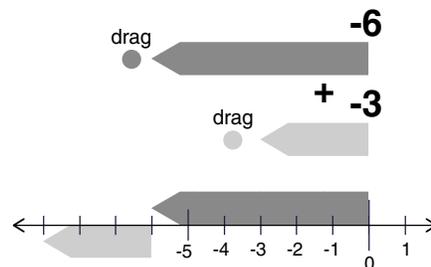
## Sketch and Investigate

All sketches referred to in this booklet can be found in **Sketchpad I Samples | Teaching Mathematics** (**Sketchpad** is the folder that contains the application itself.)

1. Open the sketch **Add Integers.gsp**.
2. Study the problem that's modeled:  $8 + 5 = 13$ . Then drag the two "drag" circles to model other addition problems. Notice how the two upper arrows relate to the two lower arrows.



- Q1** Model the problem  $-6 + -3$ . According to your sketch, what is the sum of  $-6$  and  $-3$ ?

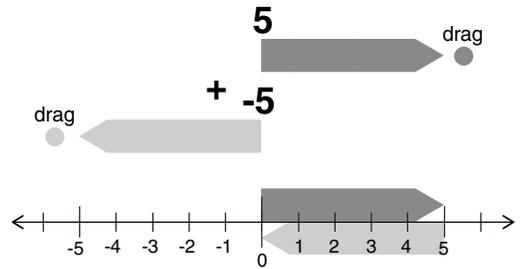


3. Model three more problems in which you add two negative numbers. Write your equations (" $-2 + -2 = -4$ ," for example) below.

**Q2** How is adding two negative numbers similar to adding two positive numbers? How is it different?

**Q3** Is it possible to add two negative numbers and get a positive sum? Explain.

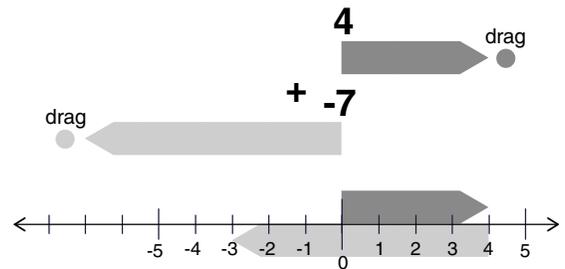
- Q4** Model the problem  $5 + -5$ .  
According to your sketch,  
what is the sum of 5 and  $-5$ ?



4. Model four more problems in which the sum is zero. Have the first number be positive in two problems and negative in two problems. Write your equations below.

- Q5** What must be true about two numbers if their sum is zero?

- Q6** Model the problem  $4 + -7$ .  
According to your sketch,  
what is the sum of 4 and  $-7$ ?



5. Model six more problems in which you add one positive number and one negative number. Have the first number be positive in three problems and negative in three. Also, make sure that some problems have positive answers and others have negative answers. Write your equations below.

- Q7** When adding a positive number and a negative number, how can you tell if the answer will be positive or negative?

**Q8** A classmate says, “Adding a positive number and a negative number seems more like subtracting.” Explain what she means.

**Q9** Fill in the blanks.

- a. The sum of a positive number and a positive number is always a \_\_\_\_\_ number.
- b. The sum of a negative number and a negative number is always a \_\_\_\_\_ number.
- c. The sum of any number and \_\_\_\_\_ is always zero.
- d. The sum of a negative number and a positive number is \_\_\_\_\_ if the positive number is larger and \_\_\_\_\_ if the negative number is larger. (“Larger” here means farther from zero.)

### Explore More

To *commute* means to travel back and forth. The *Commutative Property of Addition* basically says that addends can *commute* across an addition sign without affecting the sum.

- 1. The *Commutative Property of Addition* says that for any two numbers  $a$  and  $b$ ,  $a + b = b + a$ . In other words, order doesn't matter in addition! Model two addition problems on your sketch's number line that demonstrate this property.
  - a. Given the way addition is represented in this activity, why does the Commutative Property of Addition make sense?
  - b. Does the Commutative Property of Addition work if one or both addends are negative? Give examples to support your answer.

# Points “Lining Up” in the Plane

Name(s): \_\_\_\_\_

If you’ve seen marching bands perform at football games, you’ve probably seen the following: The band members, wandering in seemingly random directions, suddenly spell a word or form a cool picture. Can these patterns be described mathematically? In this activity, you’ll start to answer this question by exploring simple patterns of dots in the  $x$ - $y$  plane.

## Sketch and Investigate

1. Open a new sketch.

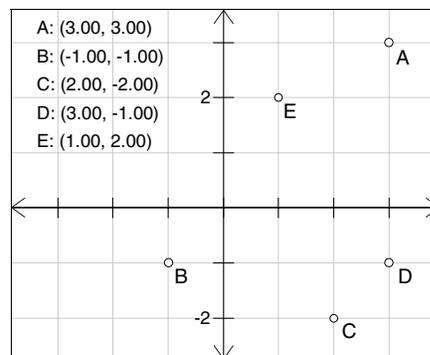
Holding down the Shift key keeps all five points selected.

2. Choose the **Point** tool from the Toolbox. Then, while holding down the Shift key, click five times in different locations (other than on the axes) to construct five new points.

To measure the coordinates of selected points, choose **Coordinates** from the Measure menu.

3. Measure the coordinates of the five selected points.

*A coordinate system appears and the coordinates of the five points are displayed.*



To hide objects, select them and choose **Hide** from the Display menu.

4. Hide the points at  $(0, 0)$  and at  $(1, 0)$ .

5. Choose **Snap Points** from the Graph menu.

*From now on, the points will only land on locations with integer coordinates.*

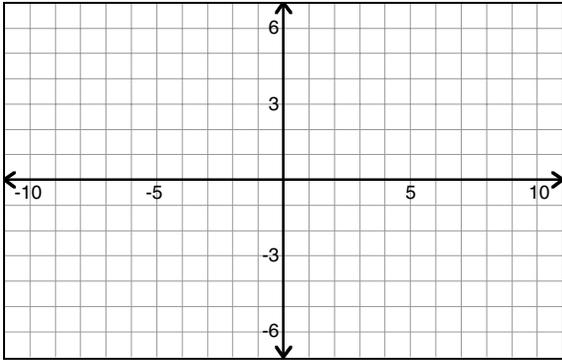
**Q1** For each problem, drag the five points to different locations that satisfy the given conditions. Then copy your solutions onto the grids on the next page.

For each point,

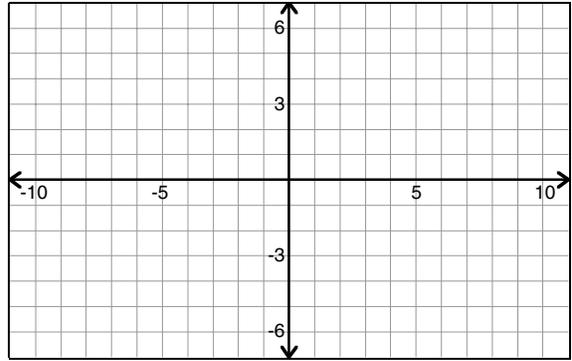
- the  $y$ -coordinate equals the  $x$ -coordinate.
- the  $y$ -coordinate is one greater than the  $x$ -coordinate.
- the  $y$ -coordinate is twice the  $x$ -coordinate.
- the  $y$ -coordinate is one greater than twice the  $x$ -coordinate.
- the  $y$ -coordinate is the opposite of the  $x$ -coordinate.
- the sum of the  $x$ - and  $y$ -coordinates is five.
- the  $y$ -coordinate is the absolute value of the  $x$ -coordinate.
- the  $y$ -coordinate is the square of the  $x$ -coordinate.

The absolute value of a number is its “positive value.”  
The absolute value of both 5 and  $-5$  is 5.

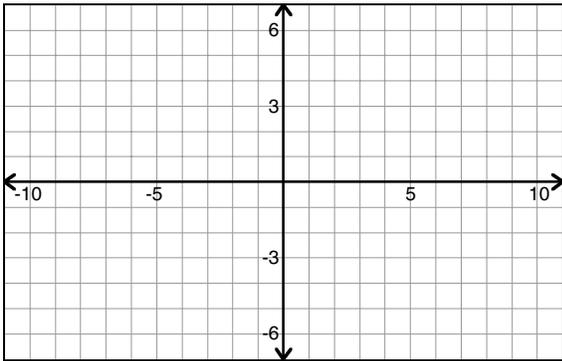
a.



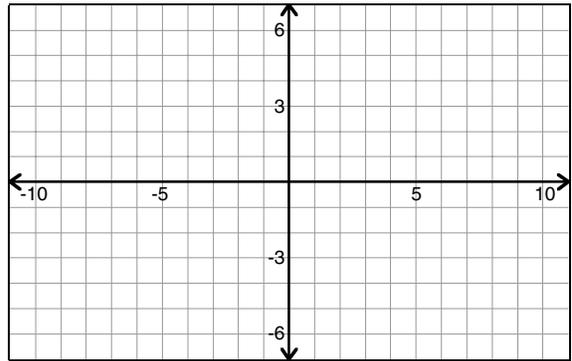
b.



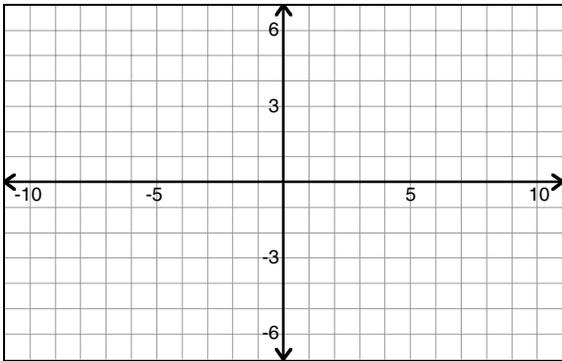
c.



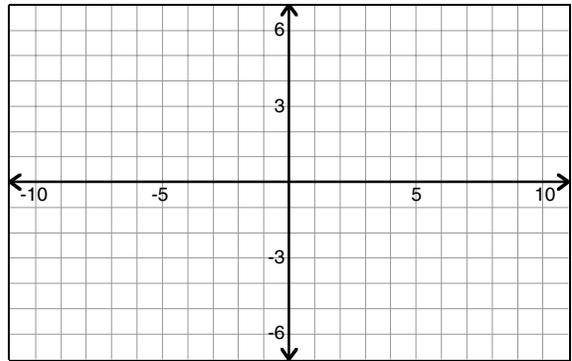
d.



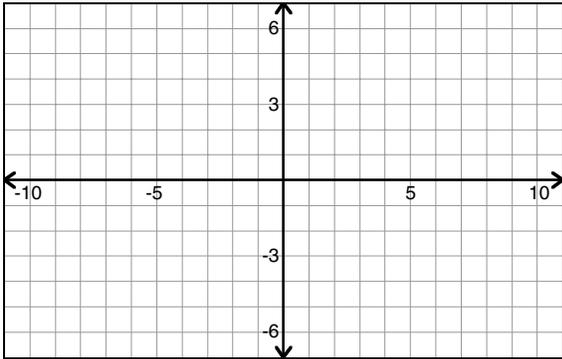
e.



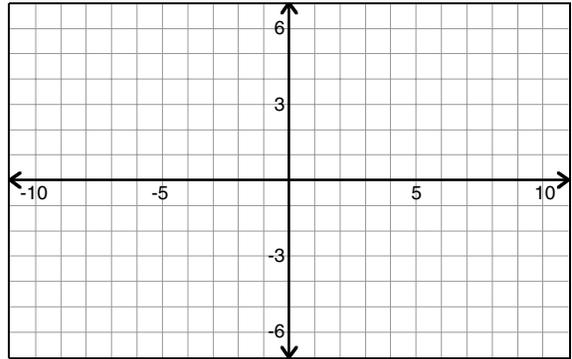
f.



g.



h.



## Backward Thinking

In Q1, you were given descriptions and asked to apply them to points. Here, we'll reverse the process and let you play detective.

All sketches referred to in this booklet can be found in **Sketchpad I Samples | Teaching Mathematics** (**Sketchpad** is the folder that contains the application itself.)

- 6. Open the sketch **Line Up.gsp**.

*You'll see a coordinate system with eight points (A through H), their coordinate measurements, and eight action buttons.*

- Q2** For each letter, press the corresponding button in the sketch. Like the members of a marching band, the points will "wander" until they form a pattern. Study the coordinates of the points in each pattern, then write a description (like the ones in Q1) for each one.

a.

b.

c.

d.

e.

f.

g.

h.

## Explore More

1. Each of the "descriptions" in this activity can be written as an equation. For example, part b of Q1 ("the  $y$ -coordinate is one greater than the  $x$ -coordinate") can be written as  $y = x + 1$ . Write an equation for each description in Q1 and Q2.
2. Add your own action buttons to those in **Line Up.gsp**, then see if your classmates can come up with descriptions or equations for your patterns. Instructions on how to do this are on page 2 of the sketch.

# Leonardo da Vinci's Proof

Name(s): \_\_\_\_\_

Leonardo da Vinci (1452–1519) was a great Italian painter, engineer, and inventor during the Renaissance. He is most famous, perhaps, for his painting *Mona Lisa*. He is also credited with the following proof of the Pythagorean theorem.

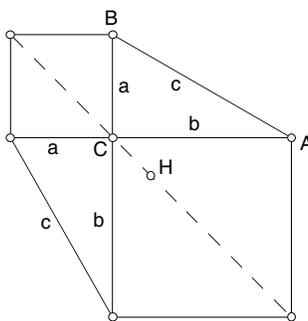
## Construct

In this figure, you don't have to construct the square on the hypotenuse.

1. Construct a right triangle and squares on the legs.
2. Connect corners of the squares to construct a second right triangle congruent to the original.
3. Construct a segment through the center of this figure, connecting far corners of the squares and passing through  $C$ .
4. Construct the midpoint,  $H$ , of this segment.

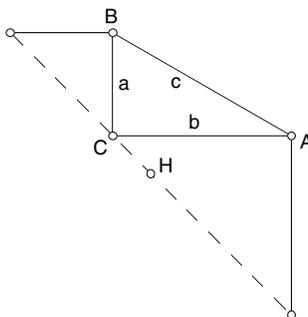
The Action Buttons submenu is in the Edit menu.

5. This segment divides the figure into mirror image halves. Select all the segments and points on one side of the center line and create a Hide/Show action button. Change its label to read "Hide reflection."



Hide reflection

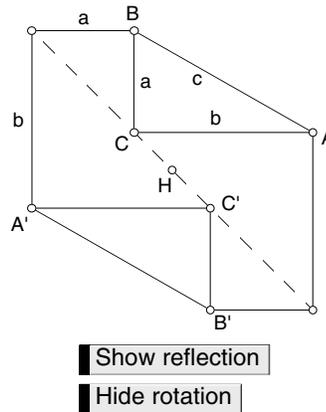
6. Press the *Hide reflection* action button. You should now see half the figure.



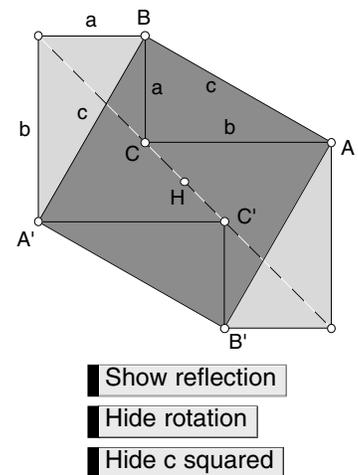
Show reflection

7. Mark  $H$  as center and rotate the entire figure (not the action buttons) by  $180^\circ$  about  $H$ .

8. Select all the objects making up the rotated half of this figure and create a Hide/Show action button. Relabel this button to read "Hide rotation," but don't hide the rotated half yet.



9. Construct  $\overline{A'B}$  and  $\overline{B'A}$ . Do you see  $c$  squared?
10. Construct the polygon interior of  $BA'B'A$  and of the two triangles adjacent to it.
11. Select  $\overline{A'B}$ ,  $\overline{B'A}$ , and the three polygon interiors and create a Hide/Show action button. Name it "Hide c squared."



### Investigate

From going through this construction, you may have a good idea of how Leonardo's proof goes. Press all the hide buttons, then play through the buttons in this sequence: *Show reflection*, *Show rotation*, *Hide reflection*, *Show c squared*. You should see the transformation from two right triangles with squares on the legs into two identical right triangles with a square on their hypotenuses. Explain to a classmate or make a presentation to the class to explain Leonardo's proof of the Pythagorean theorem.

### Prove

Leonardo's is another of those elegant proofs where the figure tells pretty much the whole story. Write a paragraph that explains why the two hexagons have equal areas and how these equal hexagons prove the Pythagorean theorem.

# The Folded Circle Construction

Name(s): \_\_\_\_\_

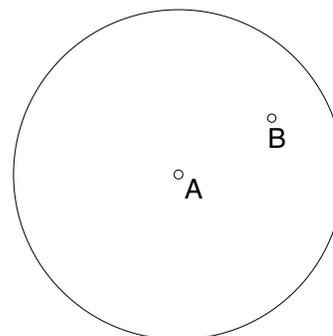
Sometimes a conic section appears in the unlikeliest of places. In this activity, you'll explore a paper-folding construction in which crease lines interact in a surprising way to form a conic.

## Constructing a Physical Model

**Preparation:** Use a compass to draw a circle with a radius of approximately three inches on a piece of wax paper or patty paper. Cut out the circle with a pair of scissors. (If you don't have these materials, you can draw the circle in Sketchpad and print it.)

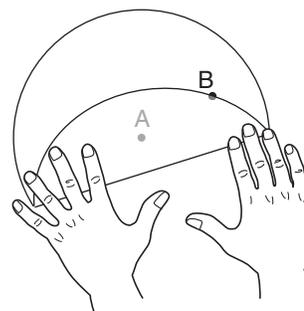
If you're working in a class, have members place  $B$  at different distances from the center. If you're working alone, do this section twice—once with  $B$  close to the center, once with  $B$  close to the edge.

1. Mark point  $A$ , the center of your circle.
2. Mark a random point  $B$  within the interior of your circle.
3. As shown below right, fold the circle so that a point on its circumference lands directly onto point  $B$ . Make a sharp crease to keep a record of this fold. Unfold the circle.
4. Fold the circle along a new crease so that a different point on the circumference lands on point  $B$ . Unfold the circle and repeat the process.
5. After you've made a dozen or so creases, examine them to see if you spot any emerging patterns.



Mathematicians would describe your set of creases as an *envelope* of creases.

6. Resume creasing your circle. Gradually, a well-outlined curve will appear. Be patient—it may take a little while.
7. Discuss what you see with your classmates and compare their folded curves to yours. If you're doing this activity alone, fold a second circle with point  $B$  in a different location.



## Questions

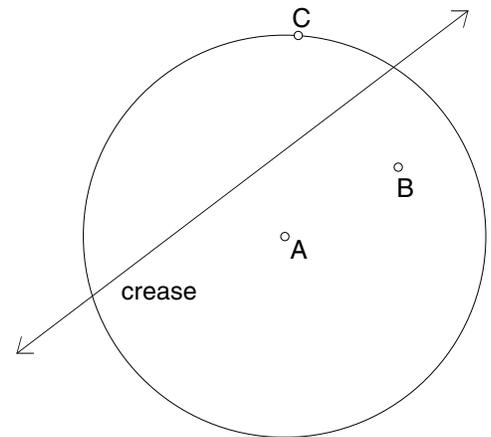
- Q1** The creases on your circle seem to form the outline of an ellipse. What appear to be its focal points?
- Q2** If you were to move point  $B$  closer to the edge of the circle and fold another curve, how do you think its shape would compare to the first curve?
- Q3** If you were to move point  $B$  closer to the center of the circle and fold another curve, how do you think its shape would compare to the first curve?

## Constructing a Sketchpad Model

Fold and unfold. Fold and unfold. Creasing your circle takes some work. Folding one or two sheets is fun, but what would happen if you wanted to continue testing different locations for point  $B$ ? You'd need to keep starting with fresh circles, folding new sets of creases.

Sketchpad can streamline your work. With just one circle and one set of creases, you can drag point  $B$  to new locations and watch the crease lines adjust themselves instantaneously.

8. Open a new sketch and use the **Compass** tool to draw a large circle with center  $A$ . Hide the circle's radius point.
9. Use the **Point** tool to draw a point  $B$  at a random spot inside the circle.
10. Construct a point  $C$  on the circle's circumference.
11. Construct the "crease" formed when point  $C$  is folded onto point  $B$ .
12. Drag point  $C$  around the circle. If you constructed your crease line correctly, it should adjust to the new locations of point  $C$ .
13. Select the crease line and choose **Trace Line** from the Display menu.
14. Drag point  $C$  around the circle to create a collection of crease lines.
15. Drag point  $B$  to a different location and then, if necessary, choose **Erase Traces** from the Display menu.
16. Drag point  $C$  around the circle to create another collection of crease lines.



If you don't want your traces to fade, be sure the Fade Traces Over Time box is unchecked on the Color panel of the Preferences dialog box.

Retracing creases for each location of point  $B$  is certainly faster than folding new circles. But we can do better. Ideally, your crease lines should relocate automatically as you drag point  $B$ . Sketchpad's powerful **Locus** command makes this possible.

17. Turn tracing off for your original crease line by selecting it and once again choosing **Trace Line** from the Display menu.
18. Now select your crease line and point  $C$ . Choose **Locus** from the Construct menu. An entire set of creases will appear: the locus of crease locations as point  $C$  moves along its path. If you drag point  $B$ , you'll see that the crease lines readjust automatically.
19. Save your sketch for possible future use. Give it a descriptive name such as **Creased Circle**.

### Questions

- Q4** How does the shape of the curve change as you move point  $B$  closer to the edge of the circle?
- Q5** How does the shape of the curve change as you move point  $B$  closer to the center of the circle?

The **Merge** and **Split** commands appear in the Edit menu.

- Q6** Select point  $B$  and the circle. Then merge point  $B$  onto the circle's circumference. Describe the crease pattern.
- Q7** Select point  $B$  and split it from the circle's circumference. Then merge it with the circle's center. Describe the crease pattern.

### Playing Detective

Each crease line on your circle touches the ellipse at exactly one point. Another way of saying this is that each crease is *tangent* to the ellipse. By engaging in some detective work, you can locate these tangency points and use them to construct just the ellipse without its creases.

All sketches referred to in this booklet can be found in **Sketchpad I Samples | Teaching Mathematics** (**Sketchpad** is the folder that contains the application itself.)

20. Open the sketch **Folded Circle.gsp**. You'll see a thick crease line and its locus already in place.
21. Drag point  $C$  and notice that the crease line remains tangent to the ellipse. The exact point of tangency lies at the intersection of two lines—the crease line and another line not shown in the figure on the next page. Construct this other line in your sketch as well as the point of tangency, point  $E$ .

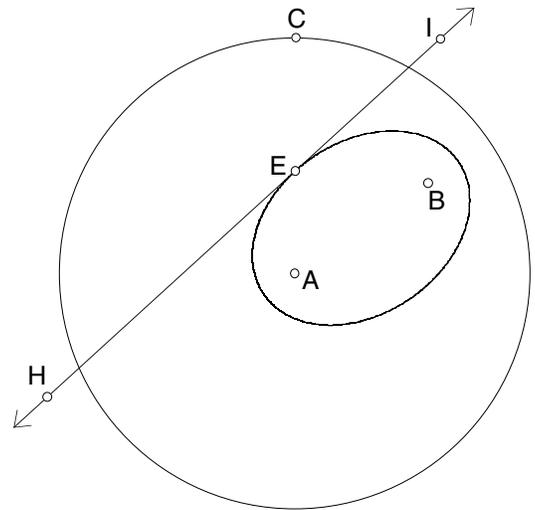
Select the locus and make its width thicker so that it's easier to see.

22. Select point  $E$  and point  $C$  and choose **Locus** from the Construct menu. If you've identified the tangency point correctly, you should see a curve appear precisely in the white space bordered by the creases.

## How to Prove It

The Folded Circle construction seems to generate ellipses. Can you prove that it does? Try developing a proof on your own, or work through the following steps and questions.

The picture at right should resemble your construction.  $\overleftrightarrow{HI}$  (the perpendicular bisector of  $\overline{CB}$ ) represents the crease formed when point  $C$  is folded onto point  $B$ . Point  $E$  sits on the curve itself.



23. Add segments  $\overline{CB}$ ,  $\overline{BE}$ , and  $\overline{AC}$  to the picture.
24. Label the intersection of  $\overline{CB}$  with the crease line as point  $D$ .

## Questions

**Q8** Use a triangle congruence theorem to prove that  $\triangle BED \cong \triangle EDH$ .

**Q9**  $\overline{BE}$  is equal in length to which other segment? Why?

**Q10** Use the distance definition of an ellipse and the result from Q9 to prove that point  $E$  traces an ellipse.

*Remember:* An ellipse is the set of points such that the sum of the distances from each point to two fixed points (the foci) is constant.

## Explore More

1. When point  $B$  lies within its circle, the creases outline an ellipse. What happens when point  $B$  lies *outside* its circle?
2. Use the illustration from your ellipse proof to show that  $\angle AEH \cong \angle BED$ .

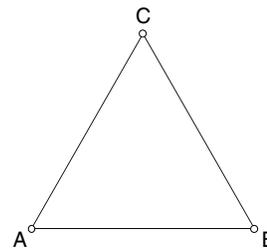
Here's an interesting consequence of this result: Imagine a pool table in the shape of an ellipse with a hole at one of its focal points. If you place a ball on the other focal point and hit it in *any* direction without spin, the ball will bounce off the side and go straight into the hole. Guaranteed!

3. The sketch **Tangent Circles.gsp** in the **Ellipse** folder shows a red circle  $c_3$  that's simultaneously tangent to circles  $c_1$  and  $c_2$ . Press the *Animate* button and observe the path of point  $C$ , the center of circle  $c_3$ . Can you prove that  $C$  traces an ellipse?

## Distances in an Equilateral Triangle

Name(s): \_\_\_\_\_

A shipwreck survivor manages to swim to a desert island. As it happens, the island closely approximates the shape of an equilateral triangle. She soon discovers that the surfing is outstanding on all three of the island's coasts. She crafts a surfboard from a fallen tree and surfs every day. Where should she build her house so that the sum of the distances from her house to all three beaches is as small as possible? (She visits each beach with equal frequency.) Before you proceed further, locate a point in the triangle at the spot where you think she should build her house.



### Conjecture

All sketches referred to in this booklet can be found in **Sketchpad I Samples | Teaching Mathematics** (Sketchpad is the folder that contains the application itself.)

1. Open the sketch **Distance.gsp**. Drag point  $P$  to experiment with your sketch.

**Q1** Press the button to show the distance sum. Drag point  $P$  around the interior of the triangle. What do you notice about the sum of the distances?

**Q2** Drag a vertex of the triangle to change the triangle's size. Again, drag point  $P$  around the interior of the triangle. What do you notice now?

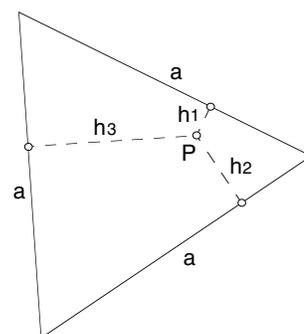
**Q3** What happens if you drag  $P$  outside the triangle?

**Q4** Organize your observations from Q1–Q3 into a conjecture. Write your conjecture using complete sentences.

### Explaining

You are no doubt convinced that the total sum of the distances from point  $P$  to all three sides of a given equilateral triangle is always constant, as long as  $P$  is an interior point. But can you explain why this is true?

Although further exploration in Sketchpad might succeed in convincing you even more fully of the truth of your conjecture, it would only confirm the conjecture's truth without providing an explanation. For example, the observation that the sun rises every morning does not explain why this is true. We have to try to explain it in terms of something else, for example, the rotation of the earth around the polar axis.



Recently, a mathematician named Mitchell Feigenbaum made some experimental discoveries in fractal geometry using a computer, just as you have used Sketchpad to discover your conjecture about a point inside an equilateral triangle. Feigenbaum's discoveries were later explained by Lanford and others. Here's what another mathematician had to say about all this:

*Lanford and other mathematicians were not trying to validate Feigenbaum's results any more than, say, Newton was trying to validate the discoveries of Kepler on the planetary orbits. In both cases the validity of the results was never in question. What was missing was the explanation. Why were the orbits ellipses? Why did they satisfy these particular relations? . . . there's a world of difference between validating and explaining.*

—M. D. Gale (1990), in  
The Mathematical Intelligencer, 12(1), 4.

### Challenge

Use another sheet of paper to try to logically explain your conjecture from Q4. After you have thought for a while and made some notes, use the balance of this activity to develop an explanation of your conjectures.

2. Press the button to show the small triangles in your sketch.

**Q5** Drag a vertex of the original triangle. Why are the three different sides all labeled  $a$ ?

**Q6** Write an expression for the area of each small triangle using  $a$  and the variables  $h_1$ ,  $h_2$ , and  $h_3$ .

**Q7** Add the three areas and simplify your expression by taking out any common factors.

**Q8** How is the sum in Q7 related to the total area of the equilateral triangle? Write an equation to show this relationship using  $A$  for the area of the equilateral triangle.

**Q9** Use your equation from Q8 to explain why the sum of the distances to all three sides of a given equilateral triangle is always constant.

**Q10** Drag  $P$  to a vertex point. How is the sum of the distances related to the altitude of the original triangle in this case?

**Q11** Explain why your explanations in Q5–Q9 would not work if the triangle were not equilateral.

### **Present Your Explanation**

Summarize your explanation of your original conjecture. You can use Q5–Q11 to help you. You might write your explanation as an argument in paragraph form or as a two-column proof. Use the back of this page, another sheet of paper, a Sketchpad sketch, or some other medium.

### **Further Exploration**

1. Construct any  $\triangle ABC$  and an arbitrary point  $P$  inside it. Where should you locate  $P$  to minimize the sum of the distances to all three sides of the triangle?
2.
  - a. Construct any rhombus and an arbitrary point  $P$  inside it. Where should you locate  $P$  to minimize the sum of the distances to all four sides of the rhombus?
  - b. Explain your observation in 2a and generalize to polygons with a similar property.
3.
  - a. Construct any parallelogram and an arbitrary point  $P$  inside it. Where should you locate  $P$  to minimize the sum of the distances to all four sides of the parallelogram?
  - b. Explain your observation in 3a and generalize to polygons with a similar property.

# Varignon Area

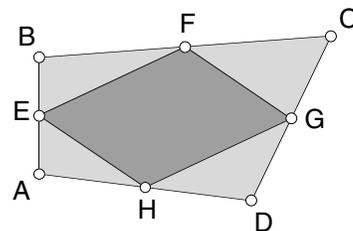
Name(s): \_\_\_\_\_

In this activity, you will compare the area of a quadrilateral to the area of another quadrilateral constructed inside it.

## Conjecture

All sketches referred to in this booklet can be found in **Sketchpad I Samples | Teaching Mathematics** (**Sketchpad** is the folder that contains the application itself.)

1. Open the sketch **Varignon.gsp** and drag vertices to investigate the shapes.



**Q1** Points  $E$ ,  $F$ ,  $G$ , and  $H$  are midpoints of the sides of quadrilateral  $ABCD$ . Describe polygon  $EFGH$ .

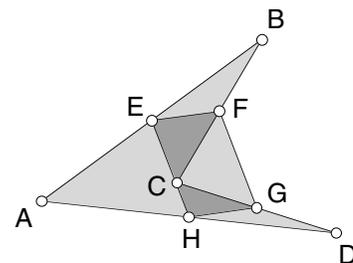
2. Press the appropriate button to show the areas of the two quadrilaterals. Drag a vertex and observe the areas.

**Q2** Describe how the areas are related. You might want to find their ratio.

**Q3** Drag any of the points  $A$ ,  $B$ ,  $C$ , and  $D$  and observe the two area measurements. Does the ratio between them change?

**Q4** Drag a vertex of  $ABCD$  until it is concave. Does this change the ratio of the areas?

**Q5** Write your discoveries so far as one or more conjectures. Use complete sentences.



**Q6** You probably can think of times when something that appeared to be always true turned out to be false on occasion. How certain are you that your conjecture is always true? Record your level of certainty on the number line and explain your choice.



## Challenge

If you believe your conjecture is always true, provide some examples to support your view and try to convince your partner or members of your group. Even better, support your conjecture with a logical explanation or a convincing proof. If you suspect your conjecture is not always true, try to supply counterexamples.

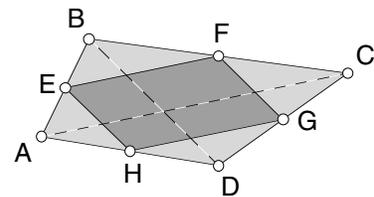
## Proving

You probably observed that quadrilateral  $EFGH$  is a parallelogram. You also probably made a conjecture that goes something like this: The area of the parallelogram formed by connecting the midpoints of the sides of a quadrilateral is half the area of the quadrilateral.

This first conjecture about quadrilateral  $EFGH$  matches a theorem of geometry that is sometimes called Varignon's theorem. Pierre Varignon was a priest and mathematician born in 1654 in Caen, France. He is known for his work with calculus and mechanics, including discoveries that relate to fluid flow and water clocks.

The next three questions will help you verify that quadrilateral  $EFGH$  is a parallelogram. If you have verified this before, skip to Q10.

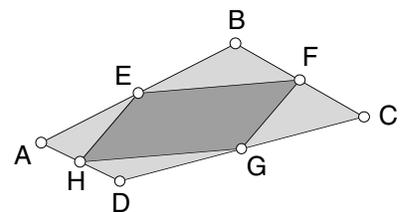
**Q7** Construct diagonal  $\overline{AC}$ . How are  $\overline{EF}$  and  $\overline{HG}$  related to  $\overline{AC}$ ? Why?



**Q8** Construct diagonal  $\overline{BD}$ . How are  $\overline{EH}$  and  $\overline{FG}$  related to  $\overline{BD}$ ? Why?

**Q9** Use Q7 and Q8 to explain why  $EFGH$  must be a parallelogram.

Work through the questions that follow for one possible explanation as to why parallelogram  $EFGH$  has half the area of quadrilateral  $ABCD$ . (If you have constructed diagonals in  $ABCD$ , it will help to delete or hide them.)

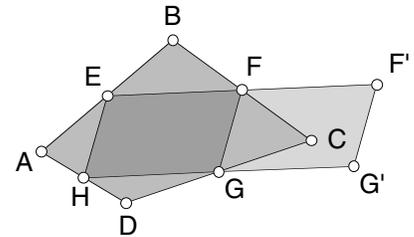


**Q10** Assume for now that  $ABCD$  is convex. One way to explain why  $ABCD$  has twice the area of  $EFGH$  is to look at the regions that are inside  $ABCD$  but not inside  $EFGH$ . Describe these regions.

**Q11** According to your conjecture, how should the total area of the regions you described in Q10 compare to the area of  $EFGH$ ?

3. Press the button to translate the midpoint quadrilateral  $EFGH$  along vector  $EF$ .

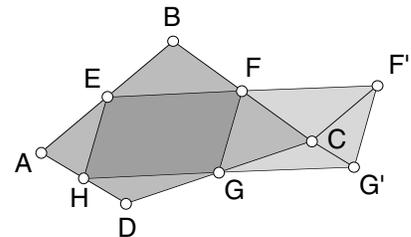
**Q12** Drag any point. How does the area of the translated quadrilateral compare to the area of  $EFGH$ ?



4. Construct  $\overline{F'C}$  and  $\overline{G'C}$ .

**Q13** How is  $\triangle EBF$  related to  $\triangle F'CF$ ?

**Q14** Explain why the relationship you described in Q13 must be true.



**Q15** How is  $\triangle HDG$  related to  $\triangle G'CG$ ?

**Q16** Explain why the relationship you described in Q15 must be true.

**Q17** How is  $\triangle AEH$  related to  $\triangle CF'G'$ ?

**Q18** Explain why the relationship you described in Q17 must be true.

**Q19** You have one more triangle to account for. Explain how this last triangle fits into your explanation.

### Present Your Proof

Create a summary of your explanations from Q10–Q19. This summary, your proof, may be in paper form or electronic form and may include a presentation sketch in Sketchpad. You may want to discuss your proof with your partner or group.

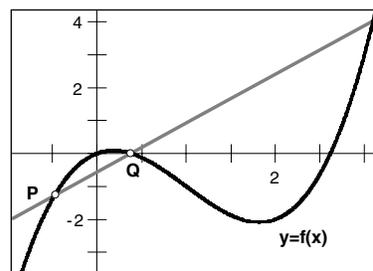
### Further Exploration

Which part of your proof does not work for concave quadrilaterals? Try to redo the proof so that it explains the concave case as well. (*Hint:* Drag point C until quadrilateral  $ABCD$  is concave.)

## Going Off on a Tangent

Name(s): \_\_\_\_\_

You can see what the average rate of change between two points on a function looks like—it's the slope of the secant line between the two points. You have also learned that as one point approaches the other, average rate approaches instantaneous rate (provided that the limit exists). But what does instantaneous rate *look like*? In this activity you will get more acquainted with the derivative and learn how to *see* it in the slope of a very special line.



### Sketch and Investigate

All sketches referred to in this booklet can be found in **Sketchpad I Samples | Teaching Mathematics** (Sketchpad is the folder that contains the application itself.)

1. Open the sketch **Tangents.gsp**.

In this sketch there is a function plotted and a line that intersects the function at a point  $P$ . This new line is called the *tangent line* because it intersects the function only once in the region near point  $P$ . Its slope is the instantaneous rate of change—or derivative—at point  $P$ :

$$\text{tangent's slope} = \text{instantaneous rate at } P = f'(x_p)$$

So how do you find this line? Let's hold off on that for a bit and look at the line's slope—the derivative—and see how it behaves. Remember, slope is the key!

Be careful here—the grid is not square!

- Q1 Move point  $P$  as close as possible to  $x = -1$ . Without using the calculator, estimate  $f'(-1)$ —the derivative of  $f$  at  $x = -1$ . (*Hint*: What's the slope of the tangent line at  $x = -1$ ?)

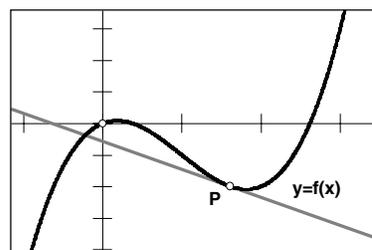
- Q2 Move point  $P$  as close as possible to  $x = 0$ . Without using the calculator, estimate  $f'(0)$ —the derivative of  $f$  at  $x = 0$ . (*Hint*: See the previous hint!)

- Q3 Move point  $P$  as close as possible to  $x = 1$ . Without using the calculator, estimate  $f'(1)$ —the derivative of  $f$  at  $x = 1$ . (Sorry, no hint this time.)

If you'd like, you can animate point  $P$  by selecting it, then choosing **Animate Point** from the Display menu.

2. Move point  $P$  back to about  $x = -1$ . Drag point  $P$  slowly along the function  $f$  from left to right. Watch the line's slope carefully so that you can answer some questions.

- Q4 For what  $x$ -values is the derivative positive? (*Hint*: When is the slope of the tangent line positive?) What can you say about the curve where the derivative is positive?



- Q5 For what  $x$ -values is the derivative negative? (*Hint*: Look at the hint in Q4 and make up your own hint.) What can you say about the curve where the derivative is negative?

- Q6 For what  $x$ -values does the derivative equal zero? What can you say about the curve where the derivative is zero?

**Q7** For what value or values of  $x$  on the interval from  $-1$  to  $3$  is the slope of the tangent line the steepest (either positive or negative)? How would you translate this question into the language of derivatives?

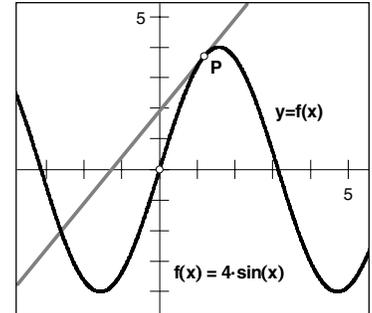
3. Go to page 2 of the document. Here the function is  $f(x) = 4 \sin(x)$ .

→ If you want to recenter your sketch, select the origin and move it to the desired location.

4. Press the *Show Zoom Tools* button and use the *x-scale* slider to change your window to go from  $-2\pi$  to  $2\pi$  on the  $x$ -axis. (You can hide the tools again by pressing the *Hide Zoom Tools* button.)

5. Move point  $P$  so that its  $x$ -coordinate is around  $x = -6$ .

6. Move point  $P$  slowly along the function to the right until you get to about  $x = 6$ . As you move the point, watch the tangent line's slope so you can answer the following questions.



**Q8** Answer Q4–Q7 for this function. Could you have relied on physical features of the graph to answer these questions quickly? (In other words, could you have answered Q4–Q7 for this function without moving point  $P$ ?)

There is an interesting relationship between how the slope is increasing or decreasing and whether the tangent line is above or below the curve. Move point  $P$  slowly from left to right again on the function, comparing the steepness of the line to its location—above or below the curve.

**Q9** When is the slope of the line increasing? Is the tangent line above or below the function when the slope is increasing?

**Q10** When is the slope of the line decreasing? Is the tangent line above or below the function when the slope is decreasing?

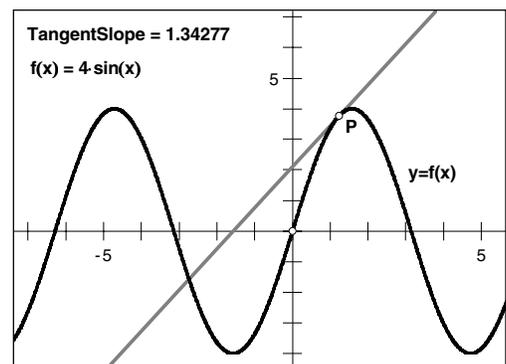
**Q11** Write your conclusion for the relationship between the slope of the tangent line and its location above or below the curve. How would you translate this into a relationship between the derivative and the function's concavity?

Let's check whether or not your conclusion is really true. The derivative is the slope of the tangent line, so an easy way to check is to calculate the slope of the line.

7. Select the tangent line and measure its slope by choosing **Slope** from the Measure menu. Label it *TangentSlope*.

8. Move point  $P$  slowly along the function again from left to right and watch the values of the measurement *TangentSlope*.

**Q12** Do your answers to Q9–Q10 hold up?



### Explore More

Each of the following functions has some interesting problems or characteristics. For each one, change the equation for  $f(x)$  by double-clicking on the expression for  $f(x)$  and entering in the new expression. Then answer the questions below. If you need to zoom in at a point, press the *Show Zoom Tools* button. Remember that  $(a, b)$  represents the point you will zoom in on. To change  $a$  or  $b$ , double-click on the parameter and enter a new value.

$$f_1(x) = |x - 2|$$

$$f_2(x) = |x^2 - 6x + 8|$$

$$f_3(x) = \sqrt{x - 1}$$

- Q1** Where does the derivative not exist for  $f_1(x)$  and why? (What happens to the tangent line at that point?)
- Q2** Answer Q1 for  $f_2(x) = |x^2 - 6x + 8|$ .
- Q3** Answer Q1 for  $f_3(x) = \sqrt{x - 1}$ .
- Q4** How is the function  $f_1(x) = |x - 2|$  different from all the others that you have looked at in this activity, including  $f_2$  and  $f_3$ ?

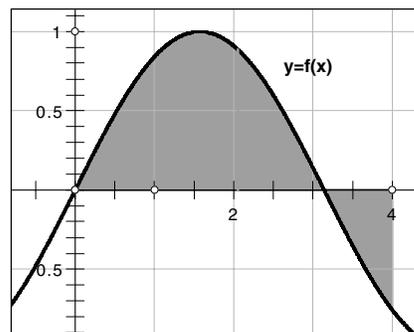
## Accumulating Area

Name(s): \_\_\_\_\_

How would you describe the shaded region shown here? You could say: The shaded region is the area between the  $x$ -axis and the curve  $f(x)$  on the interval  $0 \leq x \leq 4$ . Or, if you didn't want to use all those words, you could say: The shaded region is

$$\int_0^4 f \quad \text{or} \quad \int_0^4 f(x) dx$$

which is much faster to write!



In general, the notation  $\int_a^b f(x) dx$  represents the *signed area* between the curve  $f$  and the  $x$ -axis on the interval  $a \leq x \leq b$ . This means that the area below the  $x$ -axis is counted as negative. This activity will acquaint you with this notation, which is called the *integral*, and help you translate it into the signed area it represents.

### Sketch and Investigate

All sketches referred to in this booklet can be found in **Sketchpad I Samples I Teaching Mathematics** (Sketchpad is the folder that contains the application itself.)

1. **Open** the document **Area2.gsp**. You have a function  $f$  composed of some line segments and a semicircle connected by moveable points.

If you need to evaluate the integral  $\int_0^4 f(x) dx$ , the first step is to translate it into the language of areas. This integral stands for the area between  $f$  and the  $x$ -axis from  $x = 0$  to  $x = 4$ , as shown. This area is easy to find—you have a quarter-circle on  $0 \leq x \leq 2$  and a right triangle on  $2 \leq x \leq 4$ .

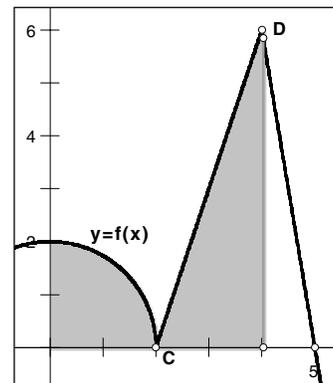
So on  $[0, 2]$  you have

$$\int_0^2 f(x) dx = 0.25\pi r^2 = 0.25\pi(2)^2 = \pi$$

and on  $[2, 4]$  you have

$$\int_2^4 f(x) dx = 0.5(\text{base})(\text{height}) = 0.5(2)(6) = 6$$

so 
$$\int_0^4 f(x) dx = 6 + \pi$$



2. To check this with the **Area** tools, press the *Show Area Tools* button.

There are three new points on the  $x$ -axis—points *start*, *finish*, and  $P$ . Points  $P$  and *start* should be at the origin. Point  $P$  will sweep out the area under the curve from point *start* to point *finish*. Point  $P$  has not moved yet, so the measurement  $\text{Area}P$  is zero.

3. Press the *Calculate Area* button to calculate the area between  $f$  and the  $x$ -axis on the interval  $[\text{start}, \text{finish}]$  and to shade in that region.

**Q1** Is the value of the measurement  $AreaP$  close to  $6 + \pi$ ? Why isn't it exactly  $6 + \pi$  or even 9.142?

**Q2** Based on the above reasoning, evaluate  $\int_{-2}^4 f(x) dx$ .

Before you move the point, check the status line to make sure you have selected the right point. If you haven't, click on the point again.

→ 4. To check your answer, move point  $start$  to point  $B$ , press the *Reset P* button, and then press the *Calculate Area* button.

**Q3** What do you think will happen to the area measurement if you switch the order of the integral, in other words, what is  $\int_4^{-2} f(x) dx$ ?

5. To check your answer, move point  $start$  to  $x = 4$  and point  $finish$  as close as you can get to  $x = -2$ , then press the *Reset P* button.

6. Choose **Erase Traces** from the Display menu and then press the *Calculate Area* button.

**Q4** What is the area between  $f$  and the  $x$ -axis from  $x = 4$  to  $x = -2$ ?

Now what happens if your function goes below the  $x$ -axis? For example, suppose you want to evaluate  $\int_4^6 f(x) dx$ .

**Q5** Translate the integral into a statement about areas.

**Q6** What familiar geometric objects make up the area you described in Q5?

The grid is shown here for comparison. It doesn't appear in the sketch.

→ **Q7** Using your familiar objects, evaluate  $\int_4^6 f(x) dx$ .  
(Hint: You can do this one quickest by thinking.)

7. Make sure point  $start$  is at  $x = 4$  and move point  $finish$  to  $x = 6$ . Press the *Reset P* button.

You can also erase traces by pressing the Esc key twice.

→ 8. Choose **Erase Traces** from the Display menu, and then press the *Calculate Area* button to check your answer. Does the result agree with your calculation?

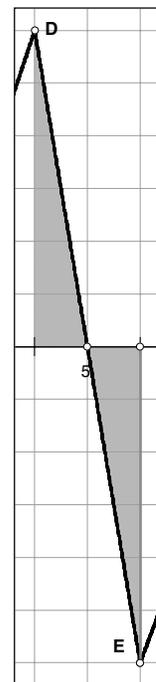
**Q8** Evaluate  $\int_{-6}^{-3} f(x) dx$  using the process in Q5–Q7 and check your answer using steps 7 and 8.

If you fix your starting point with  $x_{start} = -6$ , you can define a new function,

$A(x_p) = \int_{-6}^{x_p} f(x) dx$ , which accumulates the signed area between  $f$  and the  $x$ -axis as  $P$  moves along the  $x$ -axis.

**Q9** Why is  $A(-6) = \int_{-6}^{-6} f(x) dx = 0$ ?

**Q10** What is  $A(-3)$ ?



To get an idea of how this area function behaves as point  $P$  moves along the  $x$ -axis, you'll plot the point  $(x_p, A(x_p))$  and let Sketchpad do the work.

To turn off tracing, choose **Trace Segment** from the Display menu.

If you can't see the new point, scroll or enlarge the window until you can. To enlarge the window, press the *Show Unit Points* button, resize the window, then press the *Set Function* button.

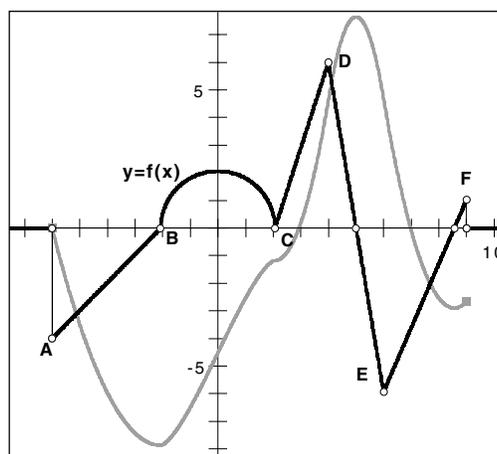
9. Move point *start* to  $x = -6$  exactly. Now move point *finish* to  $x = 9$ . The measurement *AreaP* is now the function  $A(x_p) = \int_{-6}^{x_p} f(x) dx$ .
10. Select the line segment that joins point  $P$  to the curve. Turn off tracing for the segment. Erase all traces.
11. Select measurements  $x_p$  and *AreaP* in that order and choose **Plot As (x, y)** from the Graph menu.
12. Give this new point a bright new color from the Color submenu of the Display menu. Turn on tracing for this point and label it point  $I$ .
13. Press the *Reset P* button and then the *Calculate Area* button to move point  $P$  along the  $x$ -axis and create a trace of the area function.

**Q11** Why does the area trace decrease as soon as point  $P$  moves away from point *start*?

**Q12** Why doesn't the trace become positive as soon as point  $P$  is to the right of point  $B$ ?

**Q13** What is the significance (in terms of area) of the trace's first root to the right of point *start*? of the second root?

**Q14** What is significant about the original function  $f$ 's roots? Why is this true?



14. Turn off tracing for point  $I$  and erase all traces.

Check the status line to see that point  $P$  is selected. If not, click on the point again.

15. Select points  $P$  and  $I$  and choose **Locus** from the Construct menu.

The locus you constructed should look like the trace you had above. The advantage of a locus is that if you move anything in your sketch, the locus will update itself, whereas a trace will not.

Be sure to keep points  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ , and  $F$  lined up in that order from left to right. If point  $C$  moves to the right of point  $D$ , the line segment  $CD$  will no longer exist.

There are quite a few familiar relationships between the original function  $f$  and this new locus—including the ones suggested in Q11–Q14. See if you can find some of them by trying the experiments below.

- A. Move point  $B$  (which also controls point  $C$ ) to make the radius of the semicircle larger, then smaller.
- B. Press the *Set Function* button to move point  $B$  back to  $(-2, 0)$ . Now move point  $A$  around in the plane. (Be sure to stay to the left of point  $B$ .) Try dragging point  $A$  to various places below the  $x$ -axis, and then move point  $A$  to various places above the  $x$ -axis.

C. Press the *Set Function* button to move point *A* back to  $(-6, -4)$ . Now move point *D* around in the plane. (Be sure to stay between point *C* and point *E*.) Drag point *D* to various places above the  $x$ -axis, and then drag point *D* to various places below the  $x$ -axis.

D. Follow step C with points *E* and *F*.

**Q15** List the various patterns that you found between the two functions or in the area function alone. How many patterns were you able to find? Do you have any conjectures about the relationship between the two functions?

**Explore More**

Will the area function's shape change if you move point *start* to a value other than  $x = -6$ ?

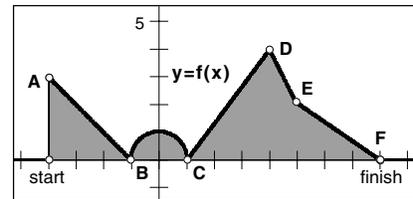
1. Select point *start* and move it along the  $x$ -axis.

**Q1** Does the area function's shape change when your starting point is shifted along the  $x$ -axis? If so, how? If not, what does change, and why?

**Q2** Write a conjecture in words for how the

two area functions  $\int_{-6}^{x_p} f(x)dx$  and

$\int_{x_{start}}^{x_p} f(x)dx$  are related.



Be sure to keep points *A*, *B*, *C*, *D*, *E*, and *F* lined up in that order from left to right. If point *C* moves to the right of point *D*, the line segment *CD* will no longer exist.

2. Make a new shape for your area function by moving one or more of points *A*, *B*, *D*, *E*, or *F*. Then move point *start* again along the  $x$ -axis.

**Q3** Does your conjecture from Q2 still hold? Write the conjecture in integral notation.

3. Fix point *start* at the origin. Move point *P* to the left of the origin but to the right of point *B*.

**Q4** The following two sentences sound good, but lead to a contradiction. Where is the error?

The semicircle is above the  $x$ -axis from the origin to point *P*, so the area is positive. Point *I*, which plots the area, is below the  $x$ -axis, so the area is negative.

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