

Dynamic Algebra Software: Bringing Functions Alive with Sketchpad®

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The behavior of functions is a fundamental unifying concept in algebra. Because variables change, it's difficult to use static media like textbook pages and chalk boards to represent the interplay between the independent and dependent variables of a function. Similarly, static media aren't suited to showing families of functions or the way in which one function can be transformed into another. Dynamic mathematics software like The Geometer's Sketchpad allows variables to vary, makes possible a variety of representations of functions, and allows functions to be manipulated and transformed.

In this session we look at a number of activities in which the power of dynamic mathematics software gives students the opportunity to vary the variables, to generate families of functions, and to transform functions.

I will be showing these activities as presentations, but each activity is designed for students to use in a computer lab (preferably with two students at each computer). Each activity available on the web includes student activity sheets (to guide students through the activity), activity notes (to provide teachers with guidance, commentary, and answers to the activity questions), and presenter notes (to facilitate the use of the activity as a whole-class presentation).

Availability of Materials

The activities and sketches used in this talk will be available on the web at http://www.keypress.com/sketchpad/general_resources/recent_talks.

Activities

From Dynagraphs¹ to Cartesian Graphs

Students make connections between symbolic and dynagraph representations of functions by matching dynagraphs with equations, and then generate a Cartesian representation by geometrically manipulating a dynagraph.

Graphing Inequalities in Two Variables

Students explore the graphs of inequalities by dragging a point on the Cartesian plane and observing where the inequality is true and where it is false. Students investigate the relationship between an inequality and the corresponding equation, and compare the graphs of different (but related) inequalities.

Parabolas in Factored Form

Students plot the graph of a quadratic function in factored form, investigate the relationship between this form and its graph, and use their observations to create functions from various descriptions of their graphs.

Stretching and Shrinking Functions

Students stretch and shrink function graphs vertically and horizontally by multiplying the function value and dividing the function argument by constants.

Unit Circle Functions

Students use a unit circle to define the trigonometric functions. They construct a point on the unit circle, measure the coordinates of the point, and graph the coordinates as a function of the arc angle from the positive x-axis to the point, producing plots of the sine and cosine functions. They also plot the slope of the line from the origin through the point, producing a plot of the tangent function.

Box and Whisker Plots

Students manipulate data points and observe the effect of their changes on a box and whisker plot. In the process they gain an intuitive understanding of the box and whisker representation of data distribution.

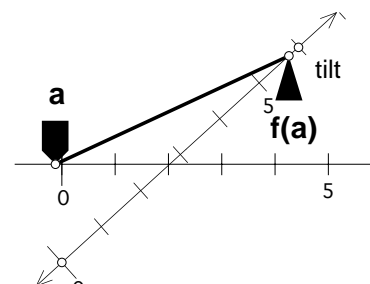
Fitting Functions to Data

Students transform functions to fit data, using a least-squares calculation to judge how good the fit is. They use several different parent functions, depending on the shape of the data.

¹ The term *dynagraph* was coined by Paul Goldenberg, Philip Lewis, and James O’Keefe in their study “Dynamic Representation and the Development of a Process Understanding of Functions” published by Education Development Center, Inc., and supported in part by a grant from the National Science Foundation.

From Dynagraphs to Cartesian Graphs (Draft)

Dynagraphs make it easy to change the input of a function and see how each input produces a corresponding output. This strength is also a weakness, because you can see only a single pair of input-output values at any time.



In this activity you'll change a dynagraph so that you can keep track of many input and output values at the same time.

FUNCTION MATCHMAKING

Start out with some “function matchmaking”—you'll match several dynagraphs to their corresponding algebraic equations.

The dynagraphs don't show any numbers, so you'll have to figure out each match by observing the output marker as you drag the input marker.

Q1 Open **Dyna To Cartesian.gsp**. The five dynagraphs on page 1 (labeled “Matching”) correspond to the equations below. Pair each dynagraph with an equation and explain how you made the match.

a. $y = x$

b. $y = -x$

c. $y = 2x$

d. $y = x^2$

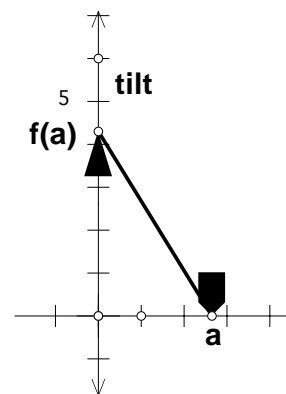
e. $y = \frac{1}{x-1} + 1$

FROM DYNAGRAPHS TO CARTESIAN GRAPHS

1. On page 2 of **Dyna To Cartesian.gsp**, you'll see a dynagraph for the function $f(x) = 2x - 1$. Drag the input marker to familiarize yourself with this function.
2. Drag the point labeled *tilt* so that the output axis is at an angle to the input axis. Drag the input marker again. Do this for a few different angles of the output axis. (You can even turn it upside down!)

Q2 When you tilt the output axis, what changes and what remains the same?

3. Press the *Make Cartesian* button and watch as the two axes of your dynagraph “morph” into the familiar x - and y -axes. Drag the input once more to convince yourself that you're still dealing with the same dynagraph, only tilted.
4. Press the *Show Perpendiculars* button to show lines through a and $f(a)$ perpendicular to the two axes.
5. Construct the intersection of the two perpendiculars.



Click on the intersection to construct the point of intersection.

6. With the new point selected, measure its coordinates by choosing **Measure | Coordinates**.
- Q3** Drag the input marker. What does the x -coordinate of the new point correspond to on the dynagraph? What does the y -coordinate correspond to?
7. Deselect all objects by clicking in blank space. Select the new point and choose **Display | Trace Intersection**. Now drag or animate the input marker and watch as P traces out the graph of $f(x)$.
- Q4** Describe the shape of the graph traced by the intersection point. Why does this shape make sense given the behavior of the dynagraph?

SIMULTANEOUS REPRESENTATION

8. Go to page 3 of **Dyna To Cartesian.gsp**.

You'll see a dynagraph and a Cartesian graph, both modeling $f(x) = 2x - 1$. Drag the input marker on the dynagraph and watch both models change simultaneously.

- Q5** A classmate says, "One Cartesian point contains the same information as two dynagraph points." Explain what she means.
9. Explore each of the following functions on the combined dynagraph/Cartesian graph. Enter the function, then drag the input marker slowly from left to right. Observe what happens to the point on the Cartesian graph as you drag.

$$f(x) = 3$$

$$f(x) = x$$

$$f(x) = -x$$

$$f(x) = x^2$$

$$f(x) = -x^2$$

$$f(x) = 5x$$

- Q6** Fill in the blanks.

When the input and output markers both move right, the Cartesian point moves _____.

When the input marker moves right and the output marker moves left, the Cartesian point moves _____.

- Q7** How does the Cartesian graph of $f(x) = 5x$ compare to that of $f(x) = x$? How does this relate to the difference between their dynagraphs?
- Q8** Compare dynagraphs and Cartesian graphs. In what ways do you think dynagraphs are better for representing functions? In what ways do you think Cartesian graphs are better?

Double-click the function equation $f(x) = 2 \cdot x - 1$ to edit it.

Objective: Students make connections between symbolic and dynagraph representations of functions by matching dynagraphs with equations, and then generate a Cartesian representation by geometrically manipulating a dynagraph.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should already have been introduced to dynagraphs.

Sketchpad Level: Easy. Students manipulate a pre-made sketch and add to it minimally.

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity (use **Dyna To Cartesian.gsp**) or Whole-Class Presentation (use **Dyna To Cartesian Present.gsp**)

The term *dynagraph* was coined by Paul Goldenberg, Philip Lewis, and James O’Keefe in their study “Dynamic Representation and the Development of a Process Understanding of Functions” published by Education Development Center, Inc., and supported in part by a grant from the National Science Foundation.

FUNCTION MATCHMAKING

Q1 Explanations will vary, but should involve the relative direction and speed of the input and output. Students can animate the input marker to make it easier to judge relative speed. Explanations may also refer to range restrictions (in d) and the existence of an asymptote (in e). Here are the matches:

- | | | |
|------|------|------|
| a. 4 | b. 2 | c. 5 |
| d. 1 | e. 3 | |

FROM DYNAGRAPHS TO CARTESIAN GRAPHS

Q2 Answers will vary. When the output axis is tilted, the two axes intersect, and there’s a change in the length and orientation of the segment connecting the two dynagraph markers. Even as these things change, the output value remains the same for any given input value, so the two markers continue to represent the same function. If $f(a) = b$ before tilting, $f(a) = b$ after tilting.

4. Consider having students construct the perpendiculars on their own instead of pressing the button.

Q3 The x -coordinate of the Cartesian point is a (the input value of the dynagraph). Similarly, the y -coordinate of

the new Cartesian point is $f(a)$ (the output value of the dynagraph).

7. After generating the trace, students can use **Construct | Locus** to create a permanent image. But it’s important that they trace first, to get a visual image of the way in which the dynagraph traces out the Cartesian graph over time.

Q4 The graph is a line with a positive slope. The fact that it’s a line corresponds to the fact that as the input marker on the dynagraph is dragged at a constant speed, the output marker moves at a constant speed. Students may make additional observations: How far the input marker goes in a given unit of time corresponds to *run*, and how far the output marker goes corresponds to *rise*. The fact that the line has a positive slope corresponds to the fact that the two markers always go the same direction (see Q6). The fact that the slope of the line is greater than 1 corresponds to the fact that the output marker moves faster than the input marker. (A slope of 1 corresponds to markers moving at the same speed; a slope between 0 and 1 corresponds to a slower output marker.)

SIMULTANEOUS REPRESENTATION

Q5 A dynagraph point contains one piece of information: its position on its axis. Thus two dynagraph points are required to show both input and output values for a function. A Cartesian point contains two pieces of information—its x -coordinate and its y -coordinate—so the Cartesian point can represent both the input and the output values at the same time.

Q6 When the input and output markers both move right, the Cartesian point moves up and to the right.

When the input marker moves right and the output marker moves left, the Cartesian point moves down and to the right.

Q7 The Cartesian graph of $f(x) = 5x$ is *steeper* than the graph of $f(x) = x$. This corresponds to the fact that the output marker moves faster for the dynagraph of $f(x) = 5x$ than it does for that of $f(x) = x$.

Q8 Student answers will vary, but should describe one advantage of each representation. Students may say that it’s easier to distinguish input values from output values on a dynagraph, and that the dynagraph allows

them to actually vary the variable. The dynagraph relates more closely to the fundamental definition of a function: For any given input value, it produces one single output value. The Cartesian graph makes the global picture more accessible: At a glance you can see the characteristics of the entire function for all input values, with no need to drag or animate anything. Finally, the dynagraph shows clearly where each point on the Cartesian graph comes from.

WHOLE-CLASS PRESENTATION

In this presentation, students make connections between the behavior of a dynagraph that represents a function and an equation representing the same function. They then see how modifying the dynagraph representation produces a Cartesian representation, and they view different functions in both representations simultaneously.

1. Open **Dyna To Cartesian Present.gsp**. Press *Animate* to put the top dynagraph into motion.

Q1 Have students guess which of the five equations on the bottom of the screen corresponds to the top dynagraph. Ask them to give arguments for their guesses, and encourage observations that relate the direction or speed of the dynagraph markers to the equations. Repeat for all five functions.
2. Page 2 contains a single dynagraph. Put it in motion.

Q2 Ask students what they think will happen if you drag the point labeled *tilt*.

Q3 With the input marker still moving, drag point *tilt* so that the bottom axis is only slightly off horizontal. Ask students to observe what has changed about the behavior of the dynagraph and what has stayed the same. Try this with several different positions of point *tilt*. The class should agree that although the direction of the output marker's motion has changed, the motion of the marker along its axis, and its numeric value, remain unchanged.
3. With the input marker in motion, press the *Make Cartesian* button.

Q4 Ask students to report what they observe about the angle of the output axis and about the origin of the output axis.
4. With the input marker in motion, press the *Show Perpendiculars* button and ask students to observe the

relationship between the dashed lines and the values of a and $f(a)$.

5. Show the intersection.

Q5 Ask students to observe the path of the intersection: its shape, where is it on the screen, and so forth.
6. Turn tracing on so that students can check their answers.
7. Consider using this page to look at one or two different functions, such as $f(x) = x^2$, or $f(x) = |x|$.
8. Page 3 contains both dynagraph and Cartesian representations. Animate the input marker a .

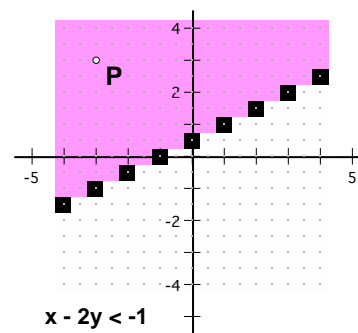
Q6 Ask students what relationships they can detect about the connections between the two models. Students may make observations about the behavior of the markers on the four axes, or about the relation between the moving point labeled $(a, f(a))$ and the motion of the dynagraph.
9. Edit the equation to change the function being displayed, and ask students how looking at a different function changes their observations about the connections.

Finish with these three questions:

- Q7** Ask students to fill in the blanks:
- When the input and output markers both move right, the Cartesian point moves _____.
- When the input marker moves right and the output marker moves left, the Cartesian point moves _____.
- Q8** How does the Cartesian graph of $f(x) = 5x$ compare to that of $f(x) = x$? How does this relate to the difference between their dynagraphs?
- Q9** Compare dynagraphs and Cartesian graphs. In what ways do you think dynagraphs are better for representing functions? In what ways do you think Cartesian graphs are better?

Graphing Inequalities in Two Variables (Draft)

Imagine yourself as a point in the coordinate plane, free to wander. In this activity, you'll travel to different locations, learning how to keep inequalities satisfied.



KEEP ME SATISFIED

1. Open **Graphing Inequalities.gsp**. Move point P around the plane, and observe how the coordinates of the point vary.

Q1 Where can you move the point so that the calculation $2x + 3$ stays greater than 5? Describe these locations.

Q2 Edit the calculation to $2y + 3$. Now where can you move the point so that the calculation of $2y + 3$ stays greater than 5? Describe these locations.

Q3 Press *Show Tracer* and move point P around again, this time anywhere you like. What happens to the tracer when the calculation $2y + 3$ is greater than 5, when it's equal to 5, and when it's less than 5?

To edit the calculation, double-click it. Click the y measurement in the sketch to enter it into the calculation.

Although the inequality $2y + 3 > 5$ has only one variable, we're still interested in all the coordinate points, which have two variables.

A graph of an inequality in two variables is the collection of all the points in the plane that *satisfy* the inequality. In this context, satisfy means "make true." For example, any time point P is in a position where $2y + 3$ is greater than 5, the inequality $2y + 3 > 5$ is true.

Q4 What points satisfy the equation $2y + 3 = 5$? How might knowing this help you describe the points that satisfy the inequality $2y + 3 > 5$?

EXPLORE

To change the calculation, double-click it and then use the Calculator. Enter values of x or y by clicking the x or y measurements in the sketch.

2. Erase the traces, hide the tracer, and change the calculation to $3x - 2y$.

Q5 Given any position of point P , if you move it straight down (without moving it left or right), the value of $3x - 2y$ increases. Explain why. What happens to the calculation if you move P to the right without moving it up or down? Why?

3. Show the tracer and move point P around the plane.

Q6 Describe the locations of point P where $3x - 2y > 5$ is true. Use your answer to Q4 to explain why any point to the right of (or below) the line $3x - 2y = 5$ is a point that satisfies the inequality $3x - 2y > 5$.

4. On page 2, drag point P and notice how it moves between closely spaced points. Show the tracer and drag P again. The tracer changes color, showing for each location of point P whether $x + 2y > -1$ is true.

Objective: Students explore the graphs of inequalities by dragging a point on the Cartesian plane and observing where the inequality is true and where it is false. Students investigate the relationship between an inequality and the corresponding equation, and compare the graphs of different (but related) inequalities.

Student Audience: Algebra 1/Algebra 2

Prerequisites: None

Sketchpad Level: Easy

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity (use **Graphing Inequalities.gsp**) or Whole-Class Presentation (use **Graphing Inequalities Present.gsp**)

Related Activities: Graphing Systems of Inequalities

KEEP ME SATISFIED

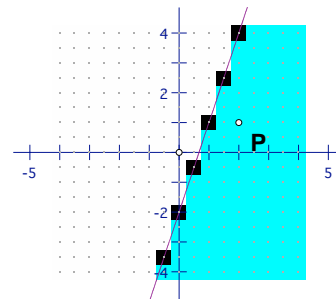
- Q1** The calculation $2x + 3$ stays greater than 5 as long as P is to the right of a vertical line through 1 on the x -axis.
- Q2** The calculation $2y + 3$ stays greater than 5 as long as P is above a horizontal line through 1 on the y -axis.
- Q3** The tracer is cyan (light blue) when the calculation is greater than 5, black when it's equal to 5, and magenta (light purple) when it's less than 5.
- Q4** The graph of $2y + 3 = 5$ is the horizontal line with a y -intercept of 1. The points satisfying $2y + 3 > 5$ are all points above that line.

EXPLORE

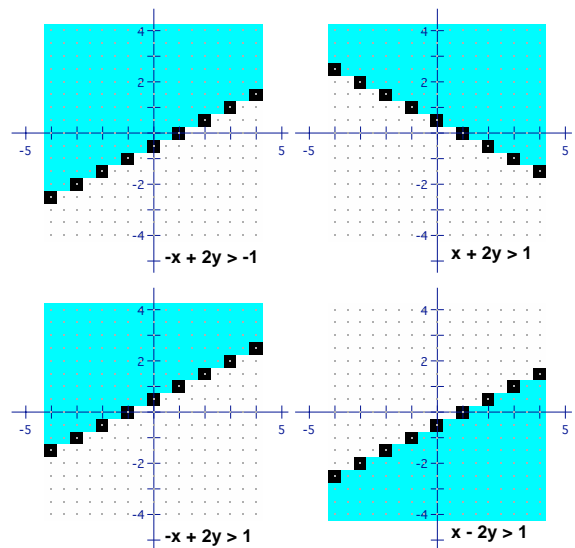
- Q5** When you move P straight down, the y -value decreases. In the expression, y is multiplied by -2 , so the value of the $-2y$ term increases when y decreases. Similarly, when you move P to the right, the x -value increases, so the value of the $3x$ term also increases.
- Q6** The statement $3x - 2y > 5$ is true when P is to the right of or below the line that is the graph of $3x - 2y = 5$. Because the value of the expression is 5 when P is on the line, and the value increases when you move P to the right or down, every point on this side of the line must correspond to an expression value greater than 5.
- Q7** In the expression $x + 2y$, the value of the expression increases when you move P to the right or up, because the first term (x) increases when you increase x by

moving P right, and the second term ($2y$) increases when you increase y by moving P up. For this reason, the statement $x + 2y > -1$ is true when P is to the right of or above the line that is the graph of $x + 2y = -1$.

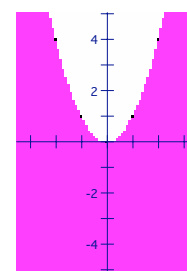
- Q8** To satisfy the inequality $3x - y > 2$, the value of the expression must be greater than the value of the expression on the line $3x - y = 2$. The x term has a positive coefficient, so P must be to the right of the line; the y term has a negative coefficient, so P must be below the line. All points to the right of and below the line satisfy the inequality.



Q9



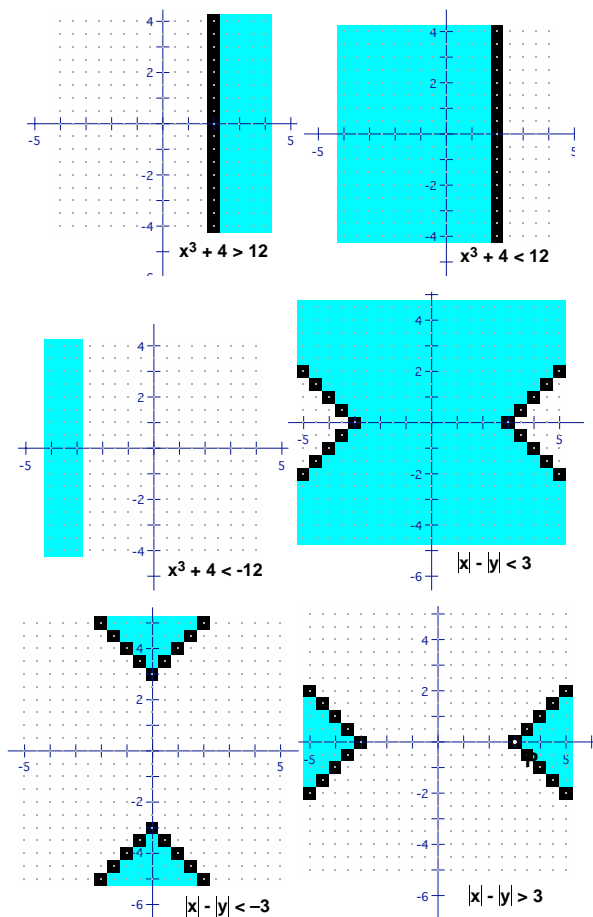
- Q10** The points that satisfy the inequality $y < x^2$ are the points below the parabola $y = x^2$.



Q11 The inequality $y > 3x - 5$ is true when y is above the line because y has a positive coefficient and is on the greater-than side of the inequality. It's true when x is left of the line because x has a positive coefficient and is on the less-than side of the inequality.

EXPLORE MORE

Q12



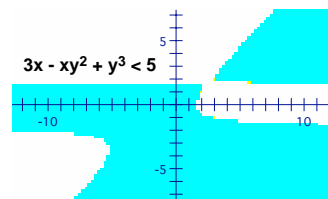
Q13 After students sketch the graphs, comparing graphs a and b and graphs d and f helps them to see that the direction of the inequality sign determines whether the cyan region or the magenta region represents the solution.

The curve(s) representing equality divide the plane into two regions. If you reverse the sign of the inequality, you change which region is the solution.

Q14 Each of these inequalities has a factored polynomial on the left side and zero on the right. To determine whether the inequality is satisfied for a given point, you only need to know the sign of the left side. Find that by determining the sign of each factor. The boundaries are

vertical lines at the roots. Every time you cross one, one of the factors changes sign, and that changes the sign of the whole polynomial. Watch out for double roots; none are used in these examples.

Q15 This inequality has an interesting shape. Students may want to adjust the grid extent to view more of it.



The related inequality $3x - xy^2 + y^3 > 5$ has the inequality sign reversed, so the solution is the complementary region (the region not colored in the diagram). The inequality $-(3x - xy^2 + y^3) < -5$ has the same shape as $3x - xy^2 + y^3 > 5$, because the direction of the inequality is reversed, and the signs of both sides have also been reversed.

If you change the inequality to an equation and solve for x in terms of y , you can graph the result. (In the Function Calculator, set the Equation form to be $x = f(y)$.) You'll need to increase the number of samples in the function plot to see the complete graph.

WHOLE-CLASS PRESENTATION

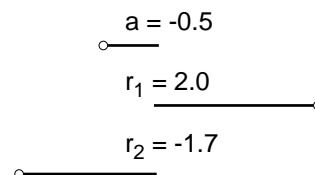
Use the sketch **Graphing Inequalities Present.gsp** to present this activity to the class. The presentation follows the steps and questions of the student activity, but is streamlined to make it easy to present. Each page of the presentation sketch has directions and buttons to help you present the activity. Use pages 1 through 7 to present the main part of the activity. Pages 8 through 10 correspond to the Explore More questions.

Parabolas in Factored Form (Draft)

If you use a parabola to model a thrown ball, you might want to know exactly where it hits the ground. Or if you model a profit function with a parabola, you might be interested in the *break-even point*—the cutoff between profitability and loss. In both cases you're interested in the *roots* of the equation—where the function's value is zero. In this activity, you'll explore a quadratic form that's based on the roots.

SKETCH AND INVESTIGATE

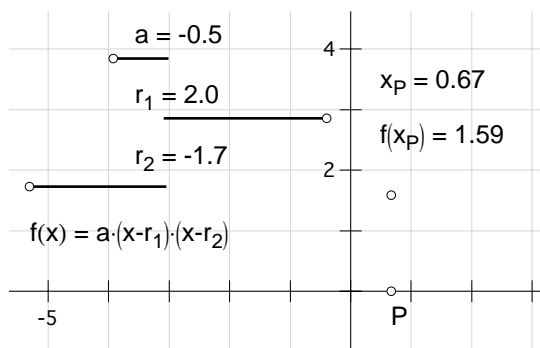
1. Open **Factored Form.gsp**. This sketch contains three sliders (a , r_1 , and r_2). Adjust each slider by dragging the point at its tip. Observe how the values change as you drag.



First you'll create a function, evaluate it for an input variable, and plot the resulting point.

2. Use the slider values to define the function $f(x) = a(x - r_1)(x - r_2)$. To do so, choose **Graph | New Function**. The New Function dialog box appears. If necessary, move it so that you can see the measurements of a , r_1 , and r_2 .
3. Enter $a*(x-r_1)*(x-r_2)$ and click OK. Sketchpad creates the function.
4. Measure the x -coordinate of point P by choosing **Measure | Abscissa**.
5. Calculate $f(x_P)$, the value of function f evaluated at x_P . To do so, choose **Measure | Calculate**. Click on the function object $f(x)$ and then on measurement x_P . Finally click OK. The value of the function appears.
6. To plot the point, select in order x_P and $f(x_P)$ and choose **Graph | Plot As (x, y)**.

To enter a , r_1 , and r_2 , click on their measurements in the sketch. To enter x , click on the x key in the dialog box.



To turn tracing on or off, select the plotted point and choose **Display | Trace Plotted Point**.

- Q1** Drag P back and forth along the x -axis to change the input variable for the function. How does the plotted point behave? Turn on tracing for the plotted point to better observe its behavior. What shape does this function trace out?

Traces are temporary in Sketchpad. Instead of using traces, you'll now make the graph permanent so you can compare the graphs of different functions.

7. Turn off tracing for the plotted point. Then choose **Display | Erase Traces** and drag P to make sure the traces no longer appear.
8. Plot a permanent graph of the function by selecting the function and choosing **Graph | Plot Function**. Drag P back and forth to make sure that the graph really corresponds to the path of the plotted point.

EXPLORING FAMILIES OF PARABOLAS

By dragging point P , you explored how x and y vary for *one particular function* with specific values of a , r_1 , and r_2 . Now you'll change the values of a , r_1 , and r_2 , which *changes the function itself*, allowing you to explore whole families of parabolas.

- Q2** Adjust slider a and describe its effect on the parabola. Discuss the effect of a 's sign (whether it's positive or negative), its magnitude (how big or small it is), and anything else that seems important.
- Q3** Dragging a appears to change all the points on the parabola but two: the x -intercepts of the parabola (the roots). Adjust all three sliders and observe the effect that each has on the x -intercepts. How is the location of the x -intercepts related to the values of the sliders?
- Q4** Adjust slider r_1 . What happens to the parabola as r_1 changes? What happens as r_2 changes?
- Q5** Adjust the sliders so that $r_1 = r_2$. Describe the resulting parabola.
- Q6** For each description below, write an equation for a parabola in factored form $f(x) = a(x - r_1)(x - r_2)$. Check your answers by adjusting the sliders.
- a. x -intercepts at $(-4, 0)$ and $(6, 0)$; vertex at $(1, -1)$
 - b. x -intercepts at $(-5, 0)$ and $(1, 0)$; contains the point $(3, 32)$
 - c. x -intercepts at $(0, 0)$ and $(-3, 0)$; contains the point $(2, 3)$
 - d. same x -intercepts as $y = 2(x - 3)(x + 1)$; contains the point $(0, -3)$
 - e. same shape as $y = 2(x - 3)(x + 1)$; x -intercepts at $(-4, 0)$ and $(1, 0)$
- Q7** You throw a baseball and it flies in a parabolic path across a field. If the ball reaches its apex (highest point) 60 feet away from you, and the apex is 40 feet above the ground, how far away from you will the ball land? What is the equation of the ball's flight in factored form? (Assume that the ball starts at the point $(0, 0)$.)

Figure out the equations using pencil and paper only. Once you think you have the equation, you can use the sketch to check your result.

EXPLORE MORE

- Q8** When you have a parabola in the form $f(x) = a(x - h)^2 + k$, it's easy to find its vertex, but harder to find its roots. The opposite is true with the form $f(x) = a(x - r_1)(x - r_2)$. Explain what you do know about the vertex of parabolas in this form. Can you write an expression for the x -coordinate of the vertex in terms of r_1 and r_2 ? The y -coordinate?
9. Use the expressions you just wrote for the coordinates of the vertex to plot the vertex. If you do this properly, the plotted point will remain at the vertex regardless of how you drag the sliders. You can use Sketchpad's Calculator to calculate the x - and y -coordinates of the vertex using the expressions you found in the previous question. You can then select the two calculations and choose **Graph | Plot As (x, y)**.
- Q9** If you successfully plotted the vertex, try this: Turn on tracing for the vertex by choosing **Display | Trace Plotted Point**. Then adjust slider r_1 . What shape does the vertex trace? Can you write an equation for this curve in terms of a and r_2 ?

Objective: Students plot the graph of a quadratic function in factored form, investigate the relationship between this form and its graph, and use their observations to create functions from various descriptions of their graphs.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should have some experience with graphing quadratic equations. It's not necessary for them to be familiar already with the factored form.

Sketchpad Level: Intermediate. Students create and evaluate a function, plot and trace a point, and plot a function.

Activity Time: 40–50 minutes. The activity is shorter if you use the alternate sketch **Factored Form 2.gsp** and start on the second page of the activity.

Setting: Paired/Individual Activity (use **Factored Form.gsp**) or Whole-Class Presentation (use **Factored Form Present.gsp**)

Related Activities: Parabolas in Standard Form, Parabolas in Vertex Form, Changing Quadratic Function Forms

It's important that students understand that a parabola remains a parabola regardless of the form its equation is written in. Any parabola's equation can be written in many different forms. The forms we study are the ones that are especially simple or that convey special information.

The factored form of a parabola is different from the standard form and the vertex form in that not all parabolas can be expressed in factored form. Parabolas that don't cross or touch the x -axis cannot be written in factored form. When you summarize this activity, then, you might discuss the existence of roots and the possibility of categorizing parabolas based on whether they have 0, 1, or 2 distinct real roots. If students have studied the discriminant, this would be a good connection to make.

SKETCH AND INVESTIGATE

Q1 The plotted point follows a curved, parabolic path. The exact shape (narrow or wide, opening up or down) depends on the values of the sliders.

EXPLORING FAMILIES OF PARABOLAS

Q2 If a is positive, the parabola opens upward; if a is negative, the parabola opens downward. The larger the

absolute value of a , the narrower the parabola. The closer a is to zero, the wider the parabola.

Q3 When you drag a , the two x -intercepts remain fixed. One x -intercept exactly matches the value of slider r_1 , while the other exactly matches the value of slider r_2 .

Q4 Dragging either r slider changes the x -intercept corresponding to that slider.

Q5 A parabola with two equal roots has its vertex on the x -axis. Such roots are called *double roots*.

- Q6**
- $y = 0.04(x + 4)(x - 6)$
 - $y = 2(x + 5)(x - 1)$
 - $y = 0.3(x - 0)(x + 3)$ or $y = 0.3x(x + 3)$
 - $y = 1(x - 3)(x + 1)$ or $y = (x - 3)(x + 1)$
 - $y = 2(x + 4)(x - 1)$

It is very important that students find the equations of these parabolas using paper and pencil calculations and use Sketchpad to check their answers.

Q7 The ball will land 120 feet away. The equation is $y = (-1/90)(x - 0)(x - 120)$ or $y = (-x/90)(x - 120)$

EXPLORE MORE

Q8 Because of the symmetry parabolas exhibit, the x -coordinate of the vertex is the average of the two roots, or $(r_1 + r_2)/2$. To find the y -coordinate of the vertex, substitute $(r_1 + r_2)/2$ for x in the original equation:

$$y = a \left(\frac{r_1 + r_2}{2} - r_1 \right) \left(\frac{r_1 + r_2}{2} - r_2 \right)$$

Simplifying, you get:

$$y = \frac{-a}{4} (r_1 - r_2)^2$$

Q9 The vertex will trace out a parabola whose vertex is at $(r_2, 0)$, opens in the opposite direction of the given parabola, and is the same shape. The equation is $y = -a(x - r_2)(x - r_2)$ or $y = -a(x - r_2)^2$.

WHOLE-CLASS PRESENTATION

Use **Factored Form Present.gsp** to present this activity to the whole class, following the directions and using the buttons in the sketch.

Stretching and Shrinking Functions (Draft)

When you analyze a new function, it's easier to understand the function's behavior and graph if you can recognize it as a transformed version of a function you already know. In this activity you'll explore function transformations that involve stretching or shrinking a parent function.

MULTIPLY THE VALUE OF THE FUNCTION

1. Open **Stretching Functions.gsp**. You will see the function graph $y = f(x)$. The function itself is not important to this investigation. It's just a curve with a distinctive shape. Point P is attached to the x -axis. The measurement x_P is the x -coordinate of P .

In this section you will transform the function by multiplying the value of the function by a constant.

Enter function f by clicking the function definition on the screen, and enter the value of x_P by clicking it on the screen.

2. Choose **Measure | Calculate** and calculate $f(x_P)$.
3. Select in order x_P and $f(x_P)$. Choose **Graph | Plot As (x, y)**. A new point appears. Label it A .

Q1 Drag point P along the x -axis. What is the path of point A ?

Now you will see what happens graphically when you multiply the value of the function by a constant.

4. Using the Sketchpad Calculator again, compute the value $2 \cdot f(x_P)$. Then plot point B with coordinates $(x_P, 2 \cdot f(x_P))$.

Q2 Where is point B in relation to point A ? Is this relationship the same for all values of x ? Test this by dragging point P .

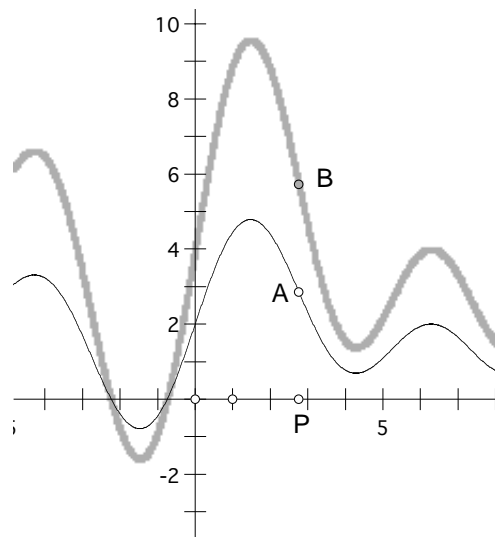
Q3 Describe the path of B as you drag point P .

5. Select B and choose **Display | Trace Plotted Point**. Drag P again to trace the path of B .

Q4 The curve you have just traced is $y = 2 \cdot f(x)$. Describe it as a transformation of $y = f(x)$.

6. Double-click the calculation $2 \cdot f(x_P)$. Change it to $0.5 \cdot f(x_P)$. Drag point P and observe the path of B again.

7. Show slider a . Change the calculation again, to make it $a \cdot f(x_P)$. Drag the slider and observe how point B behaves for



To erase the old traces, press the **Erase Traces** button at the bottom of the screen.

To construct the locus, select both points and choose **Construct | Locus**.

different values of a .

8. To avoid the need to re-create traces every time you change the multiplier a , construct the locus of point B as P moves along the x -axis.
- Q5** Use the slider to try various positive constants, some greater than 1 and some less than 1. Describe the transformation's effect on the graph when $a > 1$ and when $0 < a < 1$.
- Q6** Try negative constants. Describe the transformation's effect on the graph when $a < -1$ and when $-1 < a < 0$. What happens when $a = 1$? When $a = 0$? When $a = -1$?
- Q7** Double-click the original function to change its definition to a function of your own choice. Then try various values of a , and describe your results.

DIVIDE THE ARGUMENT OF THE FUNCTION

You just saw what happens when you multiply the function by a constant. In this section you will divide the argument (x) by a constant. You will take a shortcut by defining the transformed function directly, rather than transforming a point and constructing its locus.

To create the function, choose **Graph | Plot New Function**. To enter function f or value b into the new function, click the object in the sketch.

9. Go to page 2. This page is similar to page 1, but the slider is labeled b . Set the value of b to 1.50.
10. Create the transformed function in one step by graphing $g(x) = f(x/b)$.
- Q8** Drag slider b to divide the argument by various constants. Try several different values, both positive and negative. Describe the effect on the transformed graph when $b > 1$, when $0 < b < 1$, when $b < -1$ and when $-1 < b < 0$.
- Q9** Describe the shape of the transformed graph when $b = 1$, when $b = 0$, and when $b = -1$.

SUMMARY

11. Open page 3. The two sliders control the values a and b .
12. Define the function $g(x) = a \cdot f(x/b)$.
13. Drag each slider in turn and observe its effect.
- Q10** Why do you think the directions asked you to *multiply* the value of the function by a , but to *divide* the argument by b ?
- Q11** Describe the graph of $g(x)$ as a transformation of $f(x)$.

Objective: Students stretch and shrink function graphs vertically and horizontally by multiplying the function value and dividing the function argument by constants.

Student Audience: Algebra 2

Prerequisites: Students must be familiar with function notation and graphing.

Sketchpad Level: Intermediate. Most of the work involves editing calculations and function definitions.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use [Stretching Functions.gsp](#)) or Whole-Class Presentation (use [Stretching Functions Present.gsp](#))

MULTIPLY THE VALUE OF THE FUNCTION

- Q1** The path of A is $y = f(x)$, the function plot.
- Q2** Point B is twice as far from the x -axis as A . This relationship holds no matter where point P is.
- Q3** Since B is always twice as far from the x -axis as A , and A is always on the function graph, the path of B is a vertically stretched version of the path of A .
- Q4** The graph of $y = f(x) + 4$ is the graph of $y = f(x)$ stretched in such a way that the positive values are twice as high and the negative values are twice as low.
- Q5** For any function $f(x)$ and any constant $a > 1$, the graph of $y = a \cdot f(x)$ is the same as that of $y = f(x)$ stretched vertically by a factor of a . If $0 < a < 1$, the transformed graph shrinks toward the x -axis.
- Q6** Negative values of a flip the graph across the x -axis. When $a < -1$, the flipped graph also stretches vertically. When $-1 < a < 0$, the flipped graph shrinks vertically. When $a = 1$, the transformed graph is identical to the original. When $a = 0$ the transformed graph is flat, identical to the x -axis (a constant function with a value of zero). When $a = -1$, the transformed graph is the reflection of the original across the x -axis.
- Q7** Answers will vary, because students will choose different functions. But no matter what functions they choose, they should see the same behavior that's described for the previous question.

DIVIDE THE ARGUMENT OF THE FUNCTION

- Q8** When $b > 1$, the graph stretches horizontally, so that every point on it is farther from the y -axis. When $0 < b < 1$, the graph shrinks horizontally, so that every point on it is farther from the y -axis. When $b < -1$, the graph reflects across the y -axis and stretches horizontally. When $-1 < b < 0$, the graph reflects across the y -axis and shrinks horizontally.
- Q9** When $b = 1$, the transformed graph is identical to the original. When $b = 0$, the transformed graph does not exist, because division by zero is undefined. When $b = -1$, the transformed graph is a reflection across the y -axis.

SUMMARY

- Q10** When $a > 1$, multiplying the value of the function results in stretching vertically, because the value of the function becomes greater. When $b > 1$, dividing the argument of the function means that a greater value of the argument must be used for the same result, so this results in a horizontal stretch. (This phrasing assumes positive values, but the same principle applies if the value of the function or argument is negative.) By using multiplication in one case and division in the other, both operations result in similar behavior: a stretch when the parameter is greater than 1 and a shrink when it's between 0 and 1.
- Q11** For any function $f(x)$ and constants a and b , the graph of $y = a \cdot f(x/b)$ is the same as the graph of $y = f(x)$ transformed vertically by a stretch or shrink by a factor of a , and transformed horizontally by a stretch or shrink by a factor of b . If either a or b is negative, the graph is also reflected across the x - or y -axis.

In this presentation you will demonstrate stretches and shrinks of a graph in the form $y = a \cdot f(x/b)$ by showing the vertical and horizontal stretches and shrinks separately on pages 1 and 2, and then combine them on page 3.

1. Open **Stretching Functions Present**. Drag point P to show that it controls the measurement x_P . Drag the slider to show how it changes parameter a .
2. Press *Show A*. This point has coordinates $(x_P, f(x_P))$, so it must fall on the graph. Drag the slider so that $a = 2$.
- Q1** What will happen if you multiply the function's value by a ? If you plot the point $(x_P, a \cdot f(x_P))$, where will it fall? (It will be twice as far above the x -axis as point A .)
3. Press *Show B*. Drag P to show that B is always twice as far from the axis as A .
- Q2** What will be the path of point B as you drag P left and right? (Its path is a vertical stretch of the graph of $y = f(x)$ by a factor of 2.)
4. Select point B and choose **Display | Trace Plotted Point**. Drag point P to trace the path.
- Q3** What is the equation of this curve? ($y = 2f(x)$). Show it with other values of a by dragging slider a to a new value and then dragging P again.)
5. Go to page 2 and drag the slider so that $b = 2$.
- Q4** Now you want to plot $(x_P, f(x_P/b))$. Should you evaluate the function at point P , using the argument x_P ? (No, you need to divide x_P by 2 before evaluating.)
6. Press *Show A* to show the argument to use when evaluating the function. Press *Show B* to evaluate the function using this argument.
- Q5** Is this the point you want to plot for the new function? (No, this gives the correct value to use, but you need to plot this value at the position of P on the x -axis.)
7. Press *Show C* to plot the value of the function at the position of P . Drag P back and forth, and have students observe how the graph is stretched horizontally.
8. Page 3 has sliders for both a and b . The combined graph $y = a \cdot f(x/b)$ is plotted. Try several different combinations for the two parameters, and challenge students to describe the stretch or shrink before you reveal it in the sketch.
9. Double-click the definition of $f(x)$ to edit it. Try several different definitions. Students need to understand that the principle of stretching or shrinking a function is independent of the specific choice of the parent function.

Unit Circle Functions (Draft)

There are several different ways of defining trigonometric functions like sine and cosine. One set of definitions is based on right triangles, but right triangle definitions are limited to angles between 0 and $\pi/2$. (Recall that $\pi/2 = 90^\circ$.) In this activity you'll use a *unit circle* (a circle with a radius of exactly one unit) to define trigonometric functions for any possible angle, even beyond 2π .

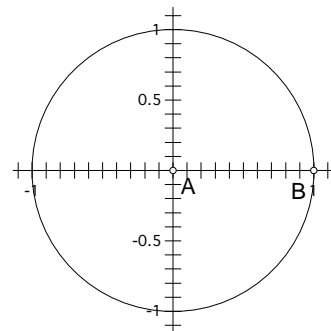
CONSTRUCT A UNIT CIRCLE

Start by creating a coordinate system, constructing a unit circle, and making some measurements.

Use the Units panel of Preferences to set units and precision, and use the Color panel to turn on trace fading.

1. In a new sketch, set the Angle Units to radians, set the Precision for slopes and ratios to thousandths, and turn on trace fading. Use the Preferences dialog box to make all three of these settings, by choosing **Edit | Preferences**.
2. Choose **Graph | Show Grid** and resize the axes (by dragging the number on one of the tick marks) so that the maximum x -value is between 6 and 7.
3. Label the origin A and the unit point B by selecting them in order and choosing **Display | Label Points**.

4. Construct a unit circle. With points A and B still selected, choose **Construct | Circle By Center+Point**.
5. Construct a point on the circle and label it C . (Be sure you don't construct it where the circle intersects one of the axes.)



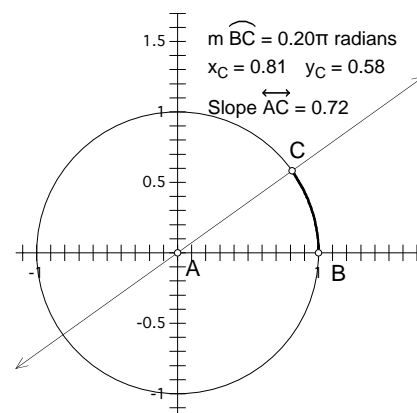
6. Measure the x - and y -coordinates of this new point separately. Choose **Measure | Abscissa (x)** and **Measure | Ordinate (y)**.
7. Construct a line through the origin and the point that you just labeled.

You'll need to select the point again after the first measurement.

Use the Line tool, or else select points A and C and choose **Construct | Line**.

8. Measure the slope of this line by choosing **Measure | Slope**.

9. On the circle, construct an arc that begins at the x -axis (at unit point B) and goes counterclockwise to point C . Make the arc thick.
10. With the arc still selected, measure its arc angle. Drag point C around the circle and observe how all four measurements behave.



To construct the arc, select the circle and points B and C in order. Then choose **Construct | Arc On Circle**. Choose **Display | Line Width | Thick** to make it thick.

Q1 What are the largest and the smallest values

you observe for each measurement? Where do you find these largest and smallest values?

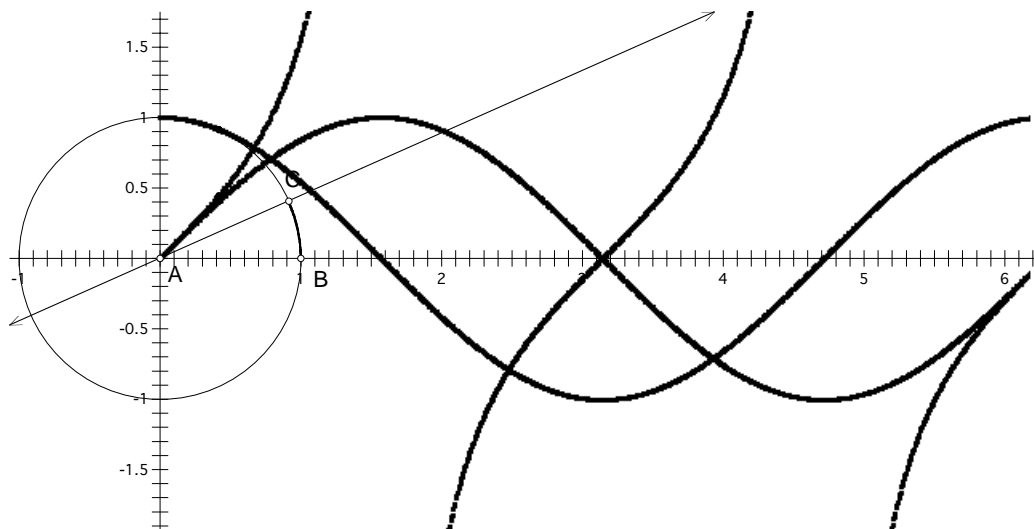
PLOT YOUR MEASUREMENTS

To explore how the measured quantities depend on the position of point C , you'll plot each measurement using the arc angle as the independent variable.

To plot the point, select in order the independent variable (the arc angle) and the dependent variable (the y -coordinate), and choose **Graph | Plot As (x, y)**.

You may want to turn off tracing for your first plotted point before looking at a second one.

11. Plot the y -coordinate of point C as a function of the arc angle. With the plotted point selected, choose **Display | Trace Plotted Point**.
- Q2** Examine the trace that appears as you drag point C around the circle. Describe its shape as you drag point C through the four quadrants. Do you recognize this graph? Which trigonometric function is this?
12. Plot the x -coordinate of point C as a function of the arc angle. Turn on tracing for this plotted point, and then drag C to observe how it behaves.
- Q3** Describe the shape of this trace as you drag point C through the four quadrants. Which trigonometric function is this?
13. Plot the slope of the line a function of the arc angle. Turn on tracing, drag C , and observe the result.



- Q4** Describe the shape of this trace as you drag C through the four quadrants. Which trigonometric function is this?
- Q5** Calculate the value of y_C/x_C . Compare this value to the value of the slope while you drag point C . What do you notice? Explain your observations.

EXPLORE MORE

- Q6** Through point B , create a tangent to the unit circle by constructing a line perpendicular to the x -axis. Construct the intersection of this tangent line with the line through points A and C , and measure the coordinate distance from the point of tangency to this intersection. How does this measurement compare with other measurements you have made? How does this measurement help explain the name of one of the trigonometric functions?

The unit circle allows you to use angles beyond $\pi/2$, but by using an arc, you are still limited to positive angles less than 2π . To use angles outside this domain, change point C so it's a rotated image of point B .

14. Create an angle parameter by choosing **Graph | New Parameter**. Name the angle θ and set its units to radians. Set its value to zero.
 15. Mark point A as the center of rotation by selecting it and choosing **Transform | Mark Center**.
 16. Rotate point B by the value of θ by selecting it and choosing **Transform | Rotate**. When the Rotate dialog box appears, click θ in the sketch to use the parameter as the angle of rotation.
 17. Split point C from the circle and merge it to the rotated image. (Select C and choose **Edit | Split Point From Circle**. Then select both C and the rotated image, and choose **Edit | Merge Points**.)
- Q7** With the parameter still selected, press the + key repeatedly to change the angle. Record at least four angles for which the slope is zero. Then use the – key to find two more angles (less than zero) for which the slope is zero.
- Q8** Find three different angles for which the y -value of the point on the circle is 1. At least one of your angles should be negative.
- Q9** Find one angle for which the x -value of the point on the circle is -1 . Then write down five more such angles, without actually trying them.
- Q10** Find one angle for which the y -value of the point on the circle is $1/2$. Then write down five more such angles.

The value of the parameter changes by $\pi/36$ (the equivalent of 5°) every time you press the + or – key. By using only these keys to change the parameter, you can keep its value at round numbers.

Objective: Students use a unit circle to define the trigonometric functions. They construct a point on the unit circle, measure the coordinates of the point, and graph the coordinates as a function of the arc angle from the positive x -axis to the point, producing plots of the sine and cosine functions. They also plot the slope of the line from the origin through the point, producing a plot of the tangent function.

Student Audience: Algebra 2/Precalculus

Prerequisites: None. You can use this activity as the first definition of the trigonometric functions, or you can use it after you've already defined the functions in a triangle.

Sketchpad Level: Intermediate. Students perform the construction and measurements themselves.

Activity Time: 35–45 minutes

Setting: Paired/Individual Activity (no sketch needed) or Whole-Class Presentation (use **Unit Circle Functions Present.gsp**)

Related Activity: Unit Circle and Right Triangle Functions

Most teachers probably define the trigonometric functions in triangles before introducing the unit circle, but others may prefer to present the unit circle definitions first. This activity supports either way of introducing these definitions.

If you are using the activity as the original definition of the functions, you should introduce it by telling students that the purpose of the activity is to define three new functions. Tell them that the function they will describe in Q2 is the sine function, the function they will describe in Q3 is the cosine function, and the function they will describe in Q4 is the tangent function.

If students have already learned the definitions of these three functions in right triangles and have seen graphs of the functions, they should be able to relate the graphs in Q2, Q3, and Q4 to the functions they already know.

In either case it's important to relate the definitions developed in this activity to the definitions in the right triangle. The activity Unit Circle and Right Triangle Functions is an excellent way to make this connection.

CONSTRUCT A UNIT CIRCLE

10. If students measure $\angle BAC$, the measurement will be shown as a value from $-\pi$ to π . By constructing an arc instead and measuring the arc angle, students end up with a measurement from 0 to 2π , and their graph shows a complete period starting at 0.

Q1 The value of x_C ranges from -1 at the left edge of the circle to $+1$ at the right edge. The value of y_C ranges from -1 at the bottom of the circle to $+1$ at the top. Answers for the slope of line AC will vary depending on how close students can drag point C to the precise bottom of the circle (where they find the smallest value, probably between -1000 and -100) and the precise top (where they find the largest value, probably between 100 and 1000). The arc angle varies from 0 at the right edge of the circle to 2π , also at the right edge of the circle.

PLOT YOUR MEASUREMENTS

Q2 The trace starts at the origin, curves up to its maximum of 1 as C leaves Quadrant I, curves back down to 0 as C leaves Quadrant II, continues to -1 as C leaves Quadrant III, and finally curves back to 0 as C reaches the end of Quadrant IV.

If students have already seen graphs of the trigonometric functions, they should recognize this as the graph of the sine function. If you are using this activity to introduce the functions, you will need to tell them that the y -coordinate on the unit circle defines the sine function.

Q3 The trace starts at $(0, 1)$, curves down to 0 as C leaves Quadrant I, continues to -1 as C leaves Quadrant II, curves back to 0 as C leaves Quadrant III, and continues on to 1 as C reaches the end of Quadrant IV.

If students have already seen graphs of the trigonometric functions, they should recognize this as the graph of the cosine function. If you are using this activity to introduce the functions, you will need to tell them that the x -coordinate on the unit circle defines the cosine function.

Q4 The trace starts at the origin, curves up and shoots off the top of the screen as C leaves Quadrant I, reappears from the bottom of the screen and curves up to 0 as C leaves Quadrant II, continues once more off the top of

the screen as C leaves Quadrant III, and reappears again at the bottom of the screen and curves back to 0 as C reaches the end of Quadrant IV.

If students have already seen graphs of the trigonometric functions, they should recognize this as the graph of the tangent function. If you are using this activity to introduce the functions, you will need to tell them that the slope from the origin to the point on the unit circle defines the tangent function.

- Q5** The calculation y_C/x_C gives the same result as the slope measurement, because it corresponds to the definition of slope: *rise/run* or $\Delta y/\Delta x$. Use students' answers to this question to make the point that the tangent function can be defined in three equivalent ways: as the slope, as y/x , or as \sin/\cos .

EXPLORE MORE

- Q6** The coordinate distance from the point of tangency to the intersection is the same as the slope measurement. Applying the slope formula to the point of intersection (x, y) and the origin $(0, 0)$ results in y/x . Because the x -coordinate along the tangent line is 1, the y -coordinate of the intersection must be equal to the slope; that is, it's equal to the value of the tangent function. Thus, the length of the tangent from the point of tangency $(1, 0)$ to this intersection is the value of the tangent function. It's this fact that originally gave the tangent function its name.

The unit circle extends the domain of the trig functions to 2π , but to go beyond 2π , students need to consider the possibility of wrapping around the unit circle more than once. The remaining three questions require them to do so, and also to consider negative angles as angles measured in the opposite (clockwise) direction.

- Q7** Angles for which the slope (tangent function) is 0 include $0, \pi, 2\pi,$ and 3π . The angle $-\pi$ is the first angle less than 0 for which the slope is 0. The general form is $n \cdot \pi$ for any integer n .
- Q8** The y -value (sine function) of the point on the circle is 1 for positive angles $\pi/2, 5\pi/2, 9\pi/2,$ and so forth. It's also 1 for negative angles $-3\pi/2, -7\pi/2,$ and so forth. The general form is $\pi/2 + 2\pi n$ for any integer n .
- Q9** The x -value (cosine function) of the point on the circle is -1 for positive angles $\pi, 3\pi, 5\pi,$ and so forth. It's also

-1 for negative angles $-\pi, -3\pi, -5\pi,$ and so forth. The general form is $\pi + 2\pi n$ for any integer n .

- Q10** The y -value (sine function) of the point on the circle is $1/2$ for positive angles $\pi/6, 5\pi/6, 13\pi/6, 17\pi/6$ and so forth. The general forms are $\pi/6 + 2\pi n$ and $5\pi/6 + 2\pi n$ for any integer.

WHOLE-CLASS PRESENTATION

Use the Presenter Notes and **Unit Circle Functions Present.gsp** to present this activity to your class.

Use this presentation to define the trigonometric functions based on the unit circle. You can use this as the fundamental introduction to these functions, or you can use it after presenting the definitions in right triangles. If students have already learned the definitions in a right triangle, you can follow this presentation with the presentation from the activity Unit Circle and Right Triangle Functions in order to relate the two ways of defining trigonometric functions.

PRESENT

1. Open **Unit Circle Functions Present.gsp**. Press the first two Show/Hide buttons to show the unit circle and the point on the unit circle.
- Q1** Ask students for the radius of the circle. (1)
2. Press the *Animate C* button to move C around the circle. Make sure that students notice the arc from B to C .
- Q2** Show the arc angle measurement, and ask students to determine the smallest and largest values for this measurement. (0 and 2π)
- Q3** Stop the animation and ask students what the arc length is. (It's the same as the arc angle, because the radius is 1.)
- Q4** Restart the animation and show the y -coordinate. Ask students to observe and determine the smallest and largest values, and where they occur. (The value of y_C ranges from -1 at the bottom of the circle to $+1$ at the top.)
- Q5** Tell students you are about to plot the y -coordinate as a dependent variable. Ask them what measurement to use as the independent variable. (arc angle BC)
3. Stop the animation and press *Show Plotted y -value* to see the plotted point. Drag C so students can verify that the plotted point matches their observations from Q4. Then restart the animation to get a smooth plot of the traced point.
- Q6** If students are already familiar with the graphs of the trig functions, ask them to identify this graph. If not, tell them that this function, defined by the y -value of a point on the unit circle, is called the sine function.
- Q7** Show the x -value and ask students to determine the smallest and largest values, and where they occur. Then show the plotted point, and either ask them to identify it or tell them that this is the definition of the cosine function.
- Q8** Repeat the same actions and questions for the slope measurement (tangent function).
- Q9** Ask students for the domain of the plots they have seen. (0 to 2π) Then go to page 2 and use the buttons to show how the domain can be extended to negative angles and to angles greater than 2π .

Box and Whisker Plots (Draft)

The *box-and-whisker plot* (sometimes just called a *box plot*) is a recent development in statistical analysis. You cannot derive any detailed information from it, but it gives you a convenient easily-understood graphical representation of the data distribution.

SKETCH AND INVESTIGATE

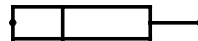
1. Open **Box and Whisker.gsp**.

The sketch contains ten data values represented as points on parallel lines. Above the points are a box with two whiskers. You can change a value by sliding its corresponding point right or left. The data are ordered and displayed on the left, but the actual numerical values are not important for this activity.

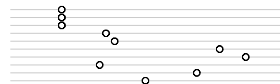
2. Before answering any of the questions, take a minute to experiment with the sketch. Drag the data points and observe the effect.

Each of the following questions suggests a special shape for the box and whiskers. In each case, state whether it is possible. If it is not possible, explain why not. If it is possible, make a rough sketch of the data points that will create that configuration, and suggest a real data set that might make this happen. The first one is done as an example.

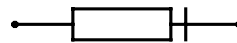
- Q1** Can one whisker have zero length?



- A1** This will occur if the lower one-fourth of the data points all have the same value. This might happen if there is a lower limit to the data range. Test a group of people to see how far they can throw a heavy weight. Those who cannot even lift the weight will all score zero.



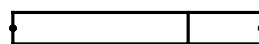
- Q2** Can the median fall outside of the box?



- Q3** Can the box have zero width?



- Q4** Can both whiskers have zero length?

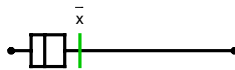


3. Press the *Show Mean* button. The mean is represented by a green bar. Normally, the mean is not shown on a box-and-whisker plot. It is appears here so that you can observe its relationship to the data distribution.

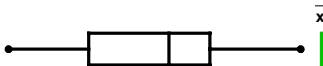
Box and Whisker Plots (Draft)

continued

Q5 Can the mean fall outside of the box?

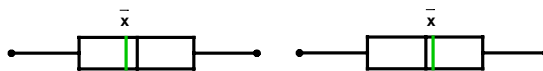


Q6 Can the mean be greater than the maximum?



Questions Q7–Q9 involve moving data points, which would not fit with any real data sample. For each question, state whether it is possible, and explain why or why not.

Q7 By moving one or more data points, can you move the mean without changing the box, the whiskers, or the median?



Q8 Can you move a single data point without changing the mean?

Q9 Can you move two data points without changing the mean?

Q10 Can you change the median by moving a single data point?

Objective: Students manipulate data points and observe the effect of their changes on a box and whisker plot. In the process they gain an intuitive understanding of the box and whisker representation of data distribution.

Prerequisites: Students should already have seen the box and whisker plot, and should understand the meaning of the five-number summary and the mean of a data sample.

Sketchpad Proficiency: Easy. Students drag points on a prepared sketch; they do no constructions.

Class Time: 40–50 minutes. The actual computer lab work can be performed in 30 minutes or less, but more time will be needed to answer the open-ended questions.

Setting: Paired/Individual Activity (use **Box and Whisker.gsp**) or Whole-Class Presentation (use **Box and Whisker Present.gsp**)

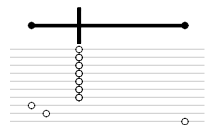
The answers to most of the questions will vary. The answers here are examples.

SKETCH AND INVESTIGATE

Q1 See A1 in the student section.

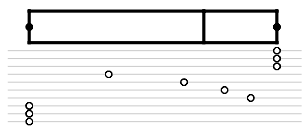
Q2 No, this is not possible. The middle half of the data must fall within the box, and the median must be somewhere in that middle half.

Q3 Yes. It will happen if the middle half of the data have the same value somewhere in the middle of the range.



Imagine a math test with only two questions, one very easy and one very difficult. Nearly all of the students would score 1, but a few outliers could score 0 or 2.

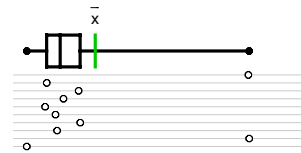
Q4 Yes. It is similar to the distribution in Q1, except that there is both a lower limit and an upper limit.



Show a photo of the cast of *Gilligan's Island* to a large group of people. Ask them to identify the characters.

Many people have no knowledge of the show, and would score zero. Of the people who have seen the show, most would have no difficulty naming all seven.

Q5 Yes. This can be caused by a fairly close grouping with some extreme outliers.



This is a situation that has been modeled in real life, in baseball labor disputes. The mean income of professional baseball players is very high. However, most professional players play in the minor leagues and earn a modest income. Relatively few outliers in the majors earn exorbitant salaries, raising the mean but having little effect on the median.

Q6 No, this is not possible. The mean must be within the range of the sample. You can prove this algebraically.

$$x_{\max} \geq x_1, x_{\max} \geq x_2, \dots, x_{\max} \geq x_{10}$$

$$10x_{\max} \geq x_1 + x_2 + \dots + x_{10}$$

$$x_{\max} \geq \frac{x_1 + x_2 + \dots + x_{10}}{10}$$

$$x_{\max} \geq \bar{x}$$

Q7 Yes. When ten data elements are ordered, the five-number summary is defined by only six of the values. The minimum is x_1 , the maximum is x_{10} , the first quartile is x_3 , the third quartile is x_7 , and the median is the mean of x_5 and x_6 . You can move the remaining four values with no effect on the box, provided they do not pass any of these above-mentioned points.

Q8 No. The mean depends on every data point, so changing a single value always affects the mean.

Q9 Yes. You can move two data points the same distance in opposite directions with no effect on the mean.

Q10 The median will change only if x_5 or x_6 is moved, or if another point is moved around one of these.

1. Open **Box and Whisker Present.gsp**.

The sketch has a set of ten data values controlled by ten points on parallel lines. A box-and-whisker summary appears above the points. The actual ordered data are in a column on the left. However, the scale is arbitrary, so you probably will have no use for the actual data.

2. Drag some of the red points to show the class how this affects the image.

- Q1** How can you make the box narrower? (Group the data more closely.)
- Q2** How can you have a narrow box but long whiskers? (Group most of the data closely, but have one outlier on each side.)

For Q3–Q8, press *Hide Box and Whiskers* before asking. Have students guide your manipulation of the data. Press *Show Box and Whiskers* after the class has reached something approaching a consensus. For each configuration, try to imagine how it could occur with real data. Some suggestions are in the activity notes.

This presentation can be particularly engaging if you give the controls to a student while you and the class direct him or her.

- Q3** How can one whisker have zero length? (Group one end of the data on the same value.)
- Q4** How can the median fall outside of the box? (This is not possible.)
- Q5** How can the box have zero width? (Group the middle part of the data on the same value.)
- Q6** Can both whiskers have zero length? (Put the three lowest on the same value and the three highest on the same value.)

3. Press the *Show Mean* button.

- Q7** How can the mean fall outside of the box? (Try closely grouped data, but with one extreme outlier.)
- Q8** How can the mean be greater than the maximum? (This is not possible.)

For the remaining questions, do not hide the box and whiskers. You will have to be able to see it as the data points are moving.

- Q9** Is it possible to move the mean without changing the box, the whiskers, or the median? (Moving any data point will change the mean, but the box and whiskers are unchanged when you move the 2nd, 4th, 7th, or 9th data points.)
- Q10** Is it possible to move one point without moving the mean? (It is not.)
- Q11** Is it possible to move two points without moving the mean? (Yes, provided they are moved the same distance in opposite directions.)

Fitting Functions to Data (Draft)

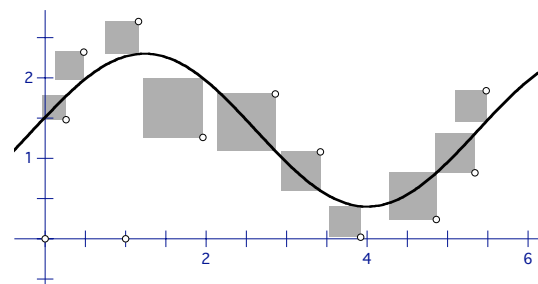
The real world doesn't often behave as cleanly as mathematical functions do. The precision and consistency of pure mathematics is one of its attractions, but also one of its challenges: It's not easy to fit the messy data of real life into the more orderly world of mathematics. In this activity you'll use a general technique for fitting functions (which are smooth and well defined) to data (which may not be).

The technique involves starting with a parent function and using translations, stretches, and shrinks to generate a transformed function that more closely fits the data. To get the best fit, you need to be able to measure how close the fit is. You will use the method of *least squares* for this purpose: You will find the vertical distances from the data points to the function, square those distances, and add them up. The smaller the sum, the better the fit. (This is the reason for the method's name.)

MAKE THE SQUARES

1. Open **Fitting Functions.gsp**.

The sketch has ten data points and four sliders. Your job is to transform a parent function using the sliders so that it fits the data points as closely as possible.



- Q1** Looking at the pattern of these data points, what parent function might you use?
2. Press *Show Parent Function* to reveal the parent function to use on this page.
3. Use the parent function and the sliders to graph the transformed function

$$g(x) = a \cdot f\left(\frac{x-h}{b}\right) + k$$

- Q2** Drag the sliders to change the transformation. Which slider adjusts the horizontal stretch/shrink? Which one adjusts the vertical translation?
4. Arrange the transformed function plot in a rough approximation of the data.

Now you're ready to measure the distances and find the sum of the squares.

5. Press and hold the **Custom** tools icon and choose **Initialize Function and Sum** from the menu that appears. To start the summation process, click this tool on the transformed function.
6. To construct the square for the first data point, choose the **Next Square** custom tool from the Custom Tools menu, and click it on the point.
7. Construct squares for each of the remaining data points.

To graph $g(x)$, choose **Graph | Plot New Function**. To enter a slider values into the formula, click it in the sketch.

You cannot see the sum yet because it is hidden.

Press and hold the **Custom** tools icon to see the menu.

8. To see the sum of the squares, choose the **Show Result** custom tool. You don't even have to click this tool; it shows the sum as soon as you choose it from the menu.

MAKE THE SQUARES SMALL (LEAST SQUARES)

As you adjust, you'll need to switch back and forth among the different sliders to get the best fit.

To make very small slider adjustments, select the point on the slider and press the right or left arrow key on the keyboard.

- Q3** Drag the sliders while you watch the *Sum of Squares* calculation, and try to make this sum as small as possible. When you're satisfied, record the slider values and the *Sum of Squares* that you used.
- Q4** Page 2 contains data to fit with a linear function. Change the parent function to $f(x) = x$. Record the slider values and the sum of squares that give the best fit.
- Q5** How many sliders do you really need in order to adjust a linear function? Explain why you don't need all four transformations in this case.
- Q6** Page 3 contains data to fit with a quadratic function. The parent function is $f(x) = x^2$. Chris adjusted the sliders to make the sum of the squares 12.10. Can you do better? Record the slider values and the sum of squares that give the best fit.
- Q7** How many sliders do you really need in order to adjust a quadratic function? Explain why you don't need all four transformations in this case.
- Q8** Page 4 contains yet more data and an exponential parent function. Fit the transformed function to the data and record your results.
- Q9** Why do you think mathematicians square the distances from the point to the function before adding them up? Why don't they just find the sum of the distances and use that to measure how good the fit is?

EXPLORE MORE

- Q10** Pages 5 and 6 contain additional data, but no functions. On these pages you must create your own parent function, transform it, and fit it to the data points. Choose your function carefully; for some data you may want to try two different parent functions to see which one fits best. (To do a second least-squares calculation in the same sketch, start over with the **Initialize Function and Sum** tool, and then use the **Next Square** tool on each data point.)
9. Collect some data of your own. The data can come from your own measurements, from a science lab, from the web, or from some other source. Plot the data in Sketchpad, and then choose an appropriate function and fit it to the data. Present the results to your group or to your entire class.

Objective: Students transform functions to fit data, using a least-squares calculation to judge how good the fit is. They use several different parent functions, depending on the shape of the data.

Student Audience: Algebra 2/Precalculus/Statistics

Prerequisites: Students must be familiar with function transformations (translations and stretches/shrinks). It's useful if they also have some experience with fitting linear functions to data.

Sketchpad Level: Intermediate. There's not much construction, but students must know how to create and graph functions. Students use custom tools to perform the least-squares calculations.

Activity Time: 35–45 minutes

Setting: Paired/Individual Activity (use **Fitting Functions.gsp**) or Whole-Class Presentation (use **Fitting Functions Present.gsp**).

MAKE THE SQUARES

- Q1** The data appear to be periodic, so $\sin x$ or $\cos x$ may be a good choice for the parent function.
- Q2** Slider b adjusts the horizontal stretch/shrink. Slider k adjusts the vertical translation.
- 5. If students are not familiar with the Custom Tools menu, show them how to press and hold the **Custom** tools icon to make it appear.

MAKE THE SQUARES SMALL (LEAST SQUARES)

Answers will vary for the questions asking students to actually fit functions. Some typical values for Q3, Q4, Q6, Q8, and Q10 are shown in the table below, but there is the possibility for considerable variation. For instance, the sin function used in Q3 will require very different values for the horizontal translation (h) if the vertical stretch (a) is negative instead of positive.

- Q5** You need only two of the sliders for a linear function. The values of a and b combine to determine the slope. The values of h and k combine to determine the intercept.
- Q7** You need only three of the sliders for a quadratic function. The values of a and b combine to determine the width. (The values of h and k determine the vertex.)

Similarly, the absolute value function needs only three sliders.

- Q9** One reason for squaring the values is to make them all positive. If you just add the deviations, a large negative deviation and a large positive one might add up to zero, even though both points are far from the graph.

Q3 (page 1)	$y = 2.05 \sin\left(\frac{x-5.20}{1.45}\right) + 0.90$
Q4 (page 2)	$y = -0.60 + 2.40$
Q6 (page 3)	$y = -\left(\frac{x-14.95}{3.20}\right)^2 + 25.25$
Q8 (page 4)	$y = 50.25 \cdot 2^{-(x-0.25)/12.60} - 0.25$
Q10 (page 5)	$y = 1.25\sqrt{x-7.90} + 0.15$
Q10 (page 6)	$y = \frac{7.5}{x-5.00} + 1.05$

WHOLE-CLASS PRESENTATION

Use the Presenter Notes and **Fitting Functions Present.gsp** to present this activity to the whole class.

Use this presentation to review function transformations and to introduce students to some of the principles of curve fitting.

MAKE THE SQUARES

1. Open **Fitting Functions Present.gsp**. The sketch has ten data points and four sliders. Tell students that their job is to transform a parent function using the sliders so that it fits the data points as closely as possible

Q1 Looking at the pattern of these data points, what sort of parent function might you use? (The data look periodic, so $\sin x$ or $\cos x$ might be a good choice.)

2. Press *Show Parent Function* to reveal the parent function for this page.
3. Press *Show Transformed Function* to create and graph the transformed function

$$g(x) = a \cdot f\left(\frac{x-h}{b}\right) + k$$

Q2 Ask students how to drag the sliders to experiment with the transformation. You're not trying to fit the data yet; you are familiarizing students with the effects of the sliders. (Students should drive this adjustment process. Don't rush them; give them time to discuss, to argue, and to think about what's happening.)

Q3 Which slider adjusts the horizontal stretch/shrink? (b)

Q4 Which slider adjusts the vertical translation? (k)

4. Adjust the sliders to arrange the transformed function plot in a rough approximation of the data.

Now you're ready to measure the distances and find the sum of the squares.

5. Use the **Initialize Function and Sum**, **Next Square**, and **Show Result** custom tools to find the sum of the squares of the deviations. Consult the directions for the student activity for details on using these tools.

Q5 Adjust the sliders to minimize the sum of the squares. Switch back and forth among the various sliders to be sure you've found a minimum result. Record the slider values.

Q6 Why do you think mathematicians square the distances from the point to the function before adding them up? Why don't they just find the sum of the distances and use that to measure how good the fit is?

The remaining pages contain a number of interesting data patterns to fit.

It's best to have a student operating the computer and taking direction from you and from other students.

To make very small slider adjustments, select the point on the slider and press the right or left arrow key on the keyboard.