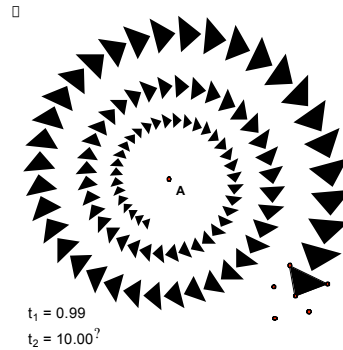


Spiral Iteration

In this activity you will use Sketchpad's **Iterate** command (on the Transform menu) to produce a spiral design. You'll also learn how to use parameters, and how to create animation action buttons for parameters.



A: An Iterated Rotation

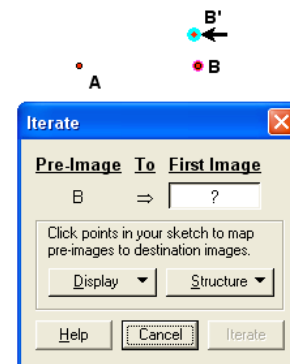
1. In a new sketch, construct points A and B . Select Point A and mark it as the center (**Transform** | **Mark Center**).
2. Select Point B and rotate it by 15 degrees. A new Point B' appears.
3. Drag each of the three points in turn.
- Q1 Does B' remain a rotated image of B , no matter how you drag the three points?
4. Select the rotated image (Point B') and rotate again by 15 degrees. Besides Point A , you now have the original Point B and two rotated images.
- Q2 How many rotated images will you need to complete an entire revolution around Point A ?

Rather than using the **Rotate** command over and over again, you'll use the **Iterate** command to do many rotations at once.

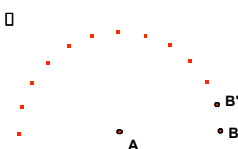
5. Select the last rotated point (B'') and delete it, so you have only A , B and B' .

The operation you want to iterate is the rotation that takes Point B to Point B' . We call Point B the *pre-image*, and Point B' the *image*. To use the **Iterate** command, you must select the pre-image (Point B) first, before choosing the command.

6. Select the pre-image (Point B) and choose **Transform** | **Iterate**. The Iterate dialog box appears.
7. In the sketch, click on the image (Point B'), as shown at the right. The label of B' appears as the First Image in the dialog box, and faint images of the first three iterated points appear in the sketch.
8. Click the dialog box's Iterate button to confirm the iteration.
9. With the iterated images selected, press the plus key (+) on the keyboard to increase the number of iterations. Continue pressing the + key until the



iterated images form a complete revolution.

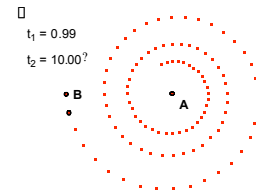
10. With the iterated images selected, press the minus key (-) on the keyboard to decrease the number of iterations. Continue pressing the - key until the iterated images form half of a complete revolution. 
11. With the iterated images selected, choose **Edit | Properties**. (Alternatively, right-click or control-click on the iterated images and choose **Properties** from the context menu that appears.) The Properties dialog appears.
12. Go to the Iteration panel of the dialog, and change the number of iterations.
- Q3 How many iterations does it take to complete an entire revolution? Why is this number different from the answer to Q2 above?
13. Select the iterated images and delete them, so you have only the three original points left.
14. Construct segment AB and segment BB' . Then construct the iteration again, following exactly the same steps as you did in steps 6 through 8 above.
- Q4 How does this second iteration look different from the original?

B: An Iterated Spiral

To make the images of Point B into a spiral, the images must not only rotate about Point A , but at the same time must become closer and closer to A . The transformation that brings one object closer to another is a dilation. In the following steps you'll add a dilation in order to create a spiral.

15. Add a new blank page to your existing document. (Choose **File | Document Options**. In the Document Options dialog box, press the Add Page pop-up, and choose **Blank Page**.)
16. Construct points A and B in the new sketch.
17. Create two parameters t_1 and t_2 in your sketch, by using the **Graph | New Parameter** command. Make the value of the first parameter 0.99 , with no units. Make the value of the second parameter 10 degrees.
18. Mark Point A as the center, and rotate Point B by the angle defined by parameter t_2 . To do this, select B and choose **Transform | Rotate**. When the Rotate dialog appears, click on parameter t_2 in the sketch, and then click Rotate.
19. Now dilate the rotated image. Select the rotated image (Point B'), and choose **Transform | Dilate**. When the Dilate dialog box appears, click on parameter t_1 in the sketch, and then click Dilate.

20. Drag-test your result, by dragging each of the points in turn. Also check the effect of the parameters by changing their values. (To change the value of a parameter, double-click it with the **Arrow** tool and type a new value.) When you're done, reset the parameters to 0.99 and 10 degrees.
21. Hide Point B' , so that only Points A , B and B'' are visible.
22. Select the pre-image point (Point B), and choose **Transform | Iterate**. Iterate pre-image B to image B'' .
23. Change the number of iterations to 100 , and observe the result. Drag Point B around, and notice the effect on the spiral.
24. Double-click on parameter t_1 and change its value to 0.995 . Also experiment with changing the value of parameter t_2 .
25. To change the parameter values more easily, select each parameter in turn and choose **Edit | Properties**. On the Parameter panel, change the Keyboard Adjustment so that the keyboard changes t_1 by 0.001 , and so that the keyboard changes t_2 by 1.0 degree. Now you can use the $+$ and $-$ keys on the keyboard to adjust these parameters easily.
- Q5 Describe the effects you observe by changing parameter t_1 . What happens when it's just slightly less than 1.0 ? What happens when it's near 0.5 ? When it's near 0.0 ? When it's greater than 1.0 ? When it's negative?
- Q6 Describe the effects you observe by changing parameter t_2 . What happens when it's close to 0 degrees? What happens when it's larger than 30 degrees? What happens when it's exactly 90 degrees? When it's 91 degrees, or 89 degrees? When it's negative?



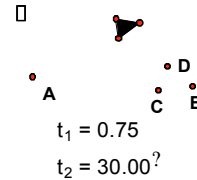
C: A Spiraling Polygon

The spiral will be more interesting if you use a polygon shape rather than just a point.

26. Add a new page to your existing document by duplicating the current page. (Choose **File | Document Options** and press the Add Page pop-up. This time choose Duplicate, and choose the current page of your document.) This new page already contains a copy of your existing construction, making it easy to use your parameters again.
27. Delete Point B . The iterated image also disappears.
28. Construct three new points: C , D and E . You'll use the images of these points to construct a triangle. (You can use more than three points if you want to use a polygon with more vertices.)



29. To make it easy to distinguish the transformed points in the next several steps, set t_1 to 0.50 and set t_2 to 30 degrees.
30. Select points C , D and E . Choose **Transform | Rotate** and rotate all three points by parameter t_2 .
31. With the rotated points still selected, choose **Transform | Dilate** and dilate them by parameter t_1 .
32. Hide the rotated points so that only the originals and the dilated points are visible.
33. Select the dilated points and choose **Construct | Triangle Interior** to make your polygon.
34. Select the original pre-image points C , D and E . Use the **Transform | Iterate** command to iterate each pre-image point to its corresponding transformed image point.
35. Set the parameters back to their original values: $t_1 = 0.99$ and $t_2 = 10$ degrees. Increase the number of iterations to 100 or more.
36. Change the shape of the spiral by dragging the points, and by changing the values of the parameters. Observe the effects.
37. Create an animation button to animate the spiral. Select parameter t_1 and choose **Edit | Action Buttons | Animation**. Set the parameter t_1 to animate from 0.90 to 1.00 by 0.01 unit every second. Try out your button and observe the results.
38. Stop the animation of parameter t_1 , and create an animation button to animate parameter t_2 . Set this button's animation to be bidirectional, from 5 degrees to 15 degrees by 2 degrees per second. Observe the results.
39. Finally, make a third button to animate both parameters at once. Select both parameters and choose **Edit | Action Buttons | Animation**. Choose settings that you think will produce an interesting animation. (You can always edit the Properties of the button to change the settings later.)



Fibonacci Iteration

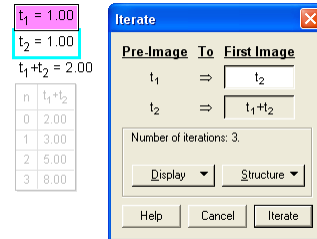
In this activity you will use Sketchpad's **Iterate** command (on the Transform menu) to generate a Fibonacci sequence.

A *Fibonacci* sequence is a sequence of numbers in which each term is the sum of the two preceding terms. The most common such sequence is the one that begins with two initial terms having the value 1: 1, 1, 2, 3, 5, 8, ... But any pair of initial terms will generate a Fibonacci sequence – the initial terms don't have to be 1 and 1.

n	t_1+t_2
0	2
1	3
2	5
3	8
4	13
5	21
6	34
7	55

A: The Fibonacci Sequence

1. In a new sketch, create two parameters t_1 and t_2 . Each parameter should have a value of 1.
2. Choose **Measure** | **Calculate**, and calculate the sum of the two parameters.
3. Select the two original parameters in the order parameters t_1 , t_2 , and then choose **Transform** | **Iterate**.
4. Iterate parameter t_1 to t_2 , and parameter t_2 to the calculation $t_1 + t_2$.
5. A faint table appears showing the results of the first three iterations. Make sure the table shows 2, 3, 5 and 8. Then click Iterate to finish the construction.
6. With the table selected, press the + key on the keyboard to increase the number of iterations. Display the first 15 iterations.



Q1 What's the 15th number in the Fibonacci sequence?

You can use this and similar iterations to investigate the characteristics of the Fibonacci sequence.

B: Investigating the Sequence

6. With the table selected, choose **Graph** | **Plot Table Data**. Click the Plot button in the dialog that appears. A graph appears showing the values in the table.
- Q2 Does the shape of the graph look familiar? What sort of function does it remind you of?

You can try using the Plot New Function command to plot a function to fit the Fibonacci sequence. If you don't know how to do this, don't worry; you'll investigate the shape of this sequence in other ways below.

In the next few steps, you'll investigate the ratio between adjacent terms of the Fibonacci sequence. To do so, you'll delete your existing table and create a new one containing more information.

7. Delete the table.
8. To investigate the ratio of adjacent terms, find the ratio of the terms you are starting with, by calculating $(t_1 + t_2) / t_2$. Change the properties of this calculation (by selecting it and using **Edit | Properties**) to show the value to the nearest hundred thousandth.
9. Construct the iteration again, by iterating parameter t_1 to t_2 , and parameter t_2 to the calculation $t_1 + t_2$.
10. Increase the number of iterations to about 10 or 15.
- Q3 What happens to the value of the ratio as the number of iterations increases?

n	t_1+t_2	$\frac{(t_1+t_2)}{t_2}$
0	2.00	2.00000
1	3.00	1.50000
2	5.00	1.66667
3	8.00	1.60000

C: Plotting Adjacent Terms

11. Add a new page to your existing document by duplicating the current page. (Choose **File | Document Options** and press the Add Page pop-up. This time choose Duplicate, and choose the current page of your document.) This new page already contains a copy of your existing construction, making it easy to use your parameters again.
12. On the new page, delete the table and the ratio calculation.
13. To get adjacent terms into the table, choose **Measure | Calculate** and create a new calculation with the value of t_2 .
14. Construct the iteration again, by iterating parameter t_1 to t_2 , and parameter t_2 to the calculation $t_1 + t_2$.
15. Increase the number of iterations to about 10.
16. Select the table and choose **Graph | Plot Table Data**. Use the Column Choice pop-ups in the dialog to plot t_2 for x and $t_1 + t_2$ for y.
17. Use the line tool to connect the last two points plotted, and measure the slope of the resulting line.
- Q4 Does this line go through any of the other plotted points? What's the value of the slope? Does this number look familiar?

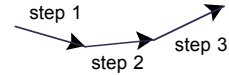
n	t_1+t_2	t_2
0	2.00	1.00
1	3.00	2.00
2	5.00	3.00
3	8.00	5.00

Iteration in Calculus

In this activity you will use Sketchpad's **Iterate** command (on the Transform menu) to approximate the anti-derivative of a function, using Euler's method.

The *anti-derivative* of a function $f(x)$ is the function $F(x)$ such that the derivative of $F(x)$ is $f(x)$ – in other words, such that $F'(x) = f(x)$. If $f(x)$ is the derivative of $F(x)$, then $F(x)$ is an anti-derivative of $f(x)$.

To use *Euler's method*, you pick a starting point and use $f(x)$ to calculate the direction in which $F(x)$ goes at that point. You take a small step in the calculated direction (step 1 in the diagram). You then calculate the direction at the new point and take another small step. By repeating this process, you can approximate the function $F(x)$ itself, even though you don't have any way to calculate or graph it directly.



In this activity, you will start with the function $f(x) = \sin(2x)$, and graph its anti-derivative – that is, graph the function whose derivative is $f(x) = \sin(2x)$.

A: Taking the First Step

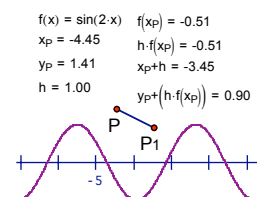
1. In a new sketch, choose **Graph | Plot New Function**, and plot the graph of $f(x) = \sin(2x)$. When you are asked if you want to switch to radians, click Yes.
2. Construct an independent Point P anywhere on the screen. This point will be the starting point for using Euler's method.
3. Measure both the abscissa and the ordinate (the x and y values) of the point.
4. Choose **Graph | New Parameter** and create a new parameter named h , with an initial value of 1.0 . The value of h will be the step size for Euler's method – the horizontal distance used for each step.

Now you must figure out the correct direction in which to step from Point P . Remember that the value of the derivative (represented by $f(x)$) is the slope of the tangent to the function. You'll figure out this slope, and then take a step in the direction determined by the slope.

5. To find the slope to use for your step, calculate the value of the derivative at Point P . In other words, calculate $f(x_p)$. Choose **Measure | Calculate**. In the Calculator, click first on the function $f(x)$, and then on the measurement x_p .
6. The value you've calculated is the slope. Think of the slope as $\Delta y / \Delta x$. You'll be using h as the value of Δx , so to find Δy you must multiply the slope you calculated in Step 5 by the value of h .
7. Use the Calculator to calculate the value of $h \cdot f(x_p)$. This will be the change in the y -value for your step.
8. Calculate the new y -value by calculating $y_p + h \cdot f(x_p)$. Calculate the new x -value by calculating $x_p + h$.

9. Plot the end point of your step by selecting the new x -value and y -value you just calculated. (Be sure to select first the x -value, and then the y -value.) Then choose **Graph } Plot As (x,y)**. Label this point P_1 .
10. The new plotted point P_1 represents the end point of the first step. Connect Point P to this end point with a segment.

You've now completed a single step of Euler's method. Fortunately, iteration makes the rest of the steps easy.



B: Taking the Rest of the Steps

11. To construct additional steps of Euler's method, you need to iterate Point P to Point P_1 . Select Point P and choose **Transform | Iterate**.
12. Use the Iterate dialog box to iterate Point P to Point P_1 .
13. Select the iterated image and choose **Edit | Properties**. Set the number of iterations (on the Iteration panel) to 300.
14. The approximation looks quite crude, because the step size is large – the value of h is 1.0. Change the value of h to 0.5.
 - Q1 What happens to the iterated image when you make h smaller?
15. The approximation will look better if you hide the point images, so that you see only the images of the segments. Select the point images and hide them.
16. Change the Parameter Properties of h so that keyboard adjustments change the value by 0.01 units.
17. Use the + and – keys to change h . Adjust h until you have a smooth result that extends over several cycles of the original function. You'll know that you have a good approximation when the shape doesn't change much as you reduce the value of h .
 - Q2 How small must h be to get a smooth and accurate approximation?
 - Q3 Describe the shape of the anti-derivative function. How do the amplitude, period and phase compare to the original function?
 - Q4 What happens to the shape of the anti-derivative when you move Point P vertically?
 - Q5 What happens to the shape of the anti-derivative when you move Point P horizontally?
 - Q6 Does the function $f(x)$ have a single anti-derivative function, or is there more than one possible anti-derivative? Explain in a paragraph.

For recent talks on iteration with Sketchpad, visit the following web site:

http://www.keypress.com/sketchpad/general_resources/recent_talks