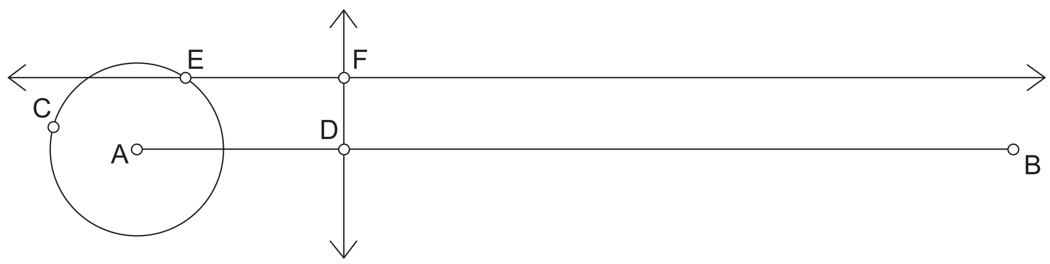
A Sine Wave Tracer

In this exploration you'll construct an animation "engine" that traces out a special curve called a *sine wave.* Variations of sine curves are the graphs of functions called *periodic functions,* functions that repeat themselves. The motion of a pendulum and ocean tides are examples of periodic functions.

SKETCH AND INVESTIGATE

1. Construct a horizontal segment *AB.*



2. Construct a circle with center *A* and radius endpoint *C.*
3. Construct point *D* on $\overline{AB}$.
4. Construct a line perpendicular to $\overline{AB}$ through point *D.*

Select point *D* and $\overline{AB}$; then, in the Construct menu, choose **Perpendicular Line.**

5. Construct point *E* on the circle.
6. Construct a line parallel to $\overline{AB}$ through point *E.*
7. Construct point *F,* the point of intersection of the vertical line through point *D* and the horizontal line through point *E.*

Q1 Drag point *D* and describe what happens to point *F.*

Don't worry, this isn't a trick question!

Q2 Drag point *E* around the circle and describe what point *F* does.

Q3 In a minute, you'll create an animation in your sketch that combines these two motions. But first try to guess what the path of point *F* will be when point *D* moves to the right along the segment at the same time that point *E* is moving around the circle. Sketch the path you imagine.

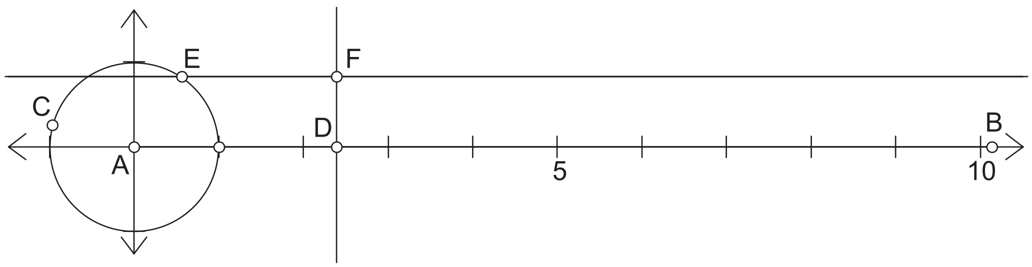
8. Make an action button that animates point *D* forward along $\overline{AB}$ and point *E* forward around the circle.

Select points *D* and *E* and choose **Edit | Action Buttons | Animation.** Choose forward in the Direction pop-up menu for point *D.*

9. Move point *D* so that it's just to the right of the circle.
10. Select point *F;* then, in the Display menu, choose **Trace Point.**
11. Press the Animation button.

Q4 Sketch the path traced by point *F*. Does the actual path resemble your guess in Q3? How is it different?

12. Select the circle; then, in the Graph menu, choose **Define Unit Circle.** You should get a graph with the origin at point *A*. Point *B* should lie on the *x*-axis. The *y*-coordinate of point *F* above $\overline{AB}$ is the value of the sine of $\angle EAD$.



Q5 If the circle has a radius of 1 grid unit, what is its circumference in grid units? (Calculate this yourself; don't use Sketchpad to measure it because Sketchpad will measure in inches or centimeters, not grid units.)

13. Measure the coordinates of point *B*.

14. Adjust the segment and the circle until you can make the curve trace back on itself instead of drawing a new curve every time. (Keep point *B* on the *x*-axis.)

Q6 What's the relationship between the *x*-coordinate of point *B* and the circumference of the circle (in grid units)? Explain why you think this is so.

Unit Circle Functions

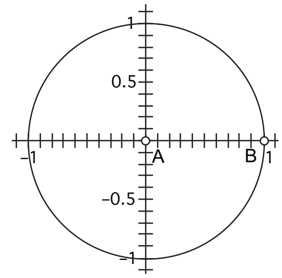
There are several different ways of defining trigonometric functions like sine and cosine. One set of definitions is based on right triangles, but right triangle definitions are limited to angles between 0 and $\pi/2$. (Recall that $\pi/2 = 90°$.) In this activity you'll use a *unit circle* (a circle with a radius of exactly one unit) to define trigonometric functions for any possible angle, even beyond 2π.

CONSTRUCT A UNIT CIRCLE

Start by creating a coordinate system, constructing a unit circle, and making some measurements.

Use the Units panel of Preferences to set units and precision, and use the Color panel to turn on trace fading.

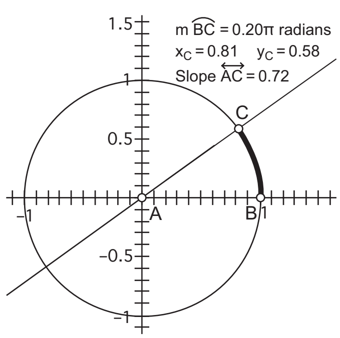
1. In a new sketch, set the Angle Units to **radians,** set the Precision for slopes and ratios to **thousandths,** and turn on trace fading. Use the Preferences dialog box to make all three of these settings, by choosing **Edit | Preferences.**
2. Choose **Graph | Show Grid** and resize the axes (by dragging the number on one of the tick marks) so that the maximum *x*-value is between 6 and 7.



3. Label the origin *A* and the unit point *B* by selecting them in order and choosing **Display | Label Points.**
4. Construct a unit circle. With points *A* and *B* still selected, choose **Construct | Circle By Center+Point.**
5. Construct a point on the circle and label it *C.* (Be sure you don't construct it where the circle intersects one of the axes.)
6. Measure the *x*- and *y*-coordinates of this new point separately. Choose **Measure | Abscissa (x)** and **Measure | Ordinate (y).**

Use the **Line** tool, or else select points *A* and *C* and choose **Construct | Line.**

7. Construct a line through the origin and the point that you just labeled.



8. Measure the slope of this line by choosing **Measure | Slope.**

To construct the arc, select the circle and points *B* and *C* in order. Then choose **Construct | Arc On Circle.** Choose **Display | Line Width | Thick** to make it thick.

9. On the circle, construct an arc that begins at the *x*-axis (at unit point *B*) and goes counter-clockwise to point *C.* Make the arc thick.
10. With the arc still selected, measure its arc angle.
    Drag point *C* around the circle and observe how all four measurements behave.

Q1 What are the largest and the smallest values you observe for each measurement? Where do you find these largest and smallest values?

PLOT YOUR MEASUREMENTS

To explore how the measured quantities depend on the position of point *C*, you'll plot each measurement using the arc angle as the independent variable.

To plot the point, select in order the independent variable (the arc angle) and the dependent variable (the *y*-coordinate), and choose **Graph | Plot As (x, y).**

11. Plot the *y*-coordinate of point *C* as a function of the arc angle. With the plotted point selected, choose **Display | Trace Plotted Point.**

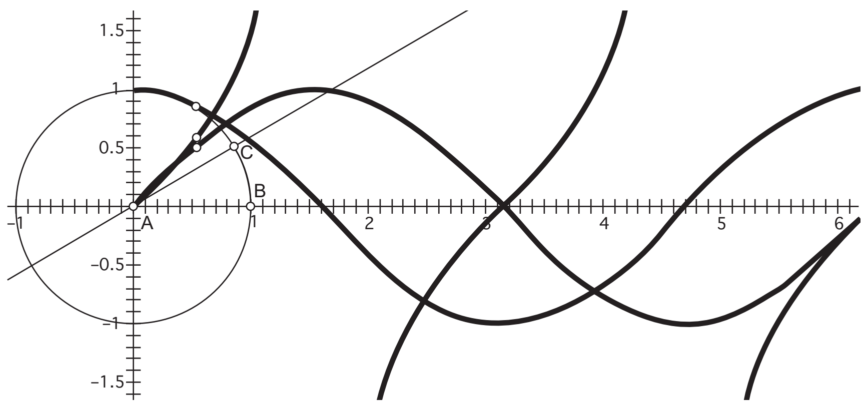
Q2 Examine the trace that appears as you drag point *C* around the circle. Describe its shape as you drag point *C* through the four quadrants. Do you recognize this graph? Which trigonometric function is this?

You may want to turn off tracing for your first plotted point before looking at the second one.

12. Plot the *x*-coordinate of point *C* as a function of the arc angle. Turn on tracing for this plotted point, and then drag *C* to observe how it behaves.

Q3 Describe the shape of this trace as you drag point *C* through the four quadrants. Which trigonometric function is this?

13. Plot the slope of the line as a function of the arc angle. Turn on tracing, drag *C*, and observe the result.



Q4 Describe the shape of this trace as you drag *C* through the four quadrants. Which trigonometric function is this?

Q5 Calculate the value of y_C/x_C. Compare this value to the value of the slope while you drag point *C*. What do you notice? Explain your observations.

EXPLORE MORE

Q6 Through point *B*, create a tangent to the unit circle by constructing a line perpendicular to the *x*-axis. Construct the intersection of this tangent line with the line through points *A* and *C*, and measure the coordinate distance from the point of tangency to this intersection. How does this measurement compare with other measurements you have made? How does this measurement help explain the name of one of the trigonometric functions?

The unit circle allows you to use angles greater than $\pi/2$, but by using an arc, you are still limited to positive angles less than 2π. To use angles outside this domain, change point *C* so it's a rotated image of point *B*.

14. Create an angle parameter by choosing **Graph | New Parameter.** Name the angle *theta* and set its units to radians.
15. Mark point *A* as the center of rotation by selecting it and choosing **Transform | Mark Center.**
16. Rotate point *B* by the value of *theta* by selecting it and choosing **Transform | Rotate.** When the Rotate dialog box appears, click *theta* in the sketch to use the parameter as the angle of rotation.
17. Split point *C* from the circle and merge it to the rotated image. (Select *C* and choose **Edit | Split Point From Circle.** Then select both *C* and the rotated image, and choose **Edit | Merge Points.**)

The value of the parameter changes by $\pi/36$ (the equivalent of 5°) every time you press the + or − key. By using only these keys to change the parameter, you can keep its value at round numbers.

Q7 Set *theta* to zero, and then press the **+** key repeatedly to change the angle. Record at least four angles for which the slope is zero. Then use the **−** key to find two more angles (less than zero) for which the slope is zero.

Q8 Find three different angles for which the *y*-value of the point on the circle is 1. At least one of your angles should be negative.

Q9 Find one angle for which the *x*-value of the point on the circle is -1. Then write down five more such angles, without actually trying them.

Q10 Find one angle for which the *y*-value of the point on the circle is 1/2. Then write down five more such angles.

Use this presentation to define the trigonometric functions based on the unit circle. You can use this as the fundamental introduction to these functions, or you can use it after presenting the definitions in right triangles.

PRESENT

1. Open **Unit Circle Functions Present.gsp.**

Q1 Ask students for the radius of the circle. (1)

2. Press the *Animate C* button to move *C* around the circle. Make sure that students notice arc *BC*.

Q2 Show the arc angle measurement, and ask students to determine the smallest and largest values for this measurement. (0 and 2π)

Do not actually measure the arc length. The measured length would not be based on the coordinate system unit.

Q3 Stop the animation and ask students what the arc length is. (It's the same as the arc angle, because the radius is 1.)

Q4 Restart the animation and show the *y*-coordinate. Ask students to observe and determine the smallest and largest values, and where they occur. (The value of y_C ranges from -1 at the bottom of the circle to $+1$ at the top.)

Q5 Tell students you are about to plot the *y*-coordinate as a dependent variable. Ask them what measurement to use as the independent variable. (arc angle *BC*)

3. Stop the animation and press *Show Plotted y-value* to see the plotted point. Drag *C* so students can verify that the plotted point matches their observations from Q4. Then restart the animation to get a smooth plot of the traced point.

Q6 What trigonometric function does this plot represent? (It is the sine function.)

4. Repeat the same actions and questions for the *x*-value (resulting in a plot of the cosine function) and the slope measurement (resulting in a plot of the tangent function).

5. Ask students for the domain of the plots they have seen. (0 to 2π) Then go to page 2 and use the buttons to show how the domain can be extended to negative angles and to angles greater than 2π.

Six Circular Functions

Trig functions are often called *circular* functions when they are defined by measurements in a unit circle.

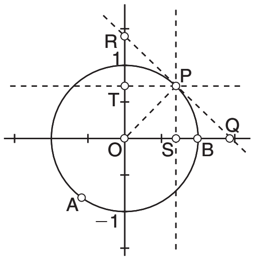
In this activity you will create a simple diagram that contains six segments corresponding to the six circular functions. You will use these segments to calculate the values of the functions and graph them.

SKETCH AND INVESTIGATE

1. In a new sketch, use the **Compass** tool to construct a circle.
2. Use the **Label** tool to label the center point *O* and the radius point *A*.
3. Define a new coordinate system using the circle as the unit circle. Hide the grid. Construct a point on the circle anywhere in Quadrant I and label it *P*. Construct the intersection of the circle with the positive *x*-axis, and label it *B*.
4. Construct the radius from *O* to *P*, and construct at *P* a tangent to the circle. Make both the radius and the tangent dashed. Label the tangent's intersection with the *x*-axis *Q* and the tangent's intersection with the *y*-axis *R*.
5. From *P* construct perpendiculars to both axes. Make both perpendiculars dashed. Label the intersection with the *x*-axis *S* and the intersection with the *y*-axis *T*.
6. Hide the tangent line and perpendiculars. Construct and label the six segments listed in the following table. Make each segment thick, and give each a different color.

To define the coordinate system, select the circle and choose **Graph | Define Unit Circle.**

To construct the tangent line, select *P* and the radius segment and choose **Construct | Perpendicular Line.**



Segment	Label	QI	→	QII	→	QIII	→	QIV	→	QI
PS	sin	+								+
PT	cos	+	0							+
PQ	tan	+	∞	−	0	+	∞	−	0	+
PR	cot	+								+
OQ	sec	+								+
OR	csc	+	1							+

7. For each segment, select its end points and measure the Coordinate Distance.

These distances correspond to the six circular functions, but because distance is always positive, you must pay attention to the behavior of the segment to determine when the corresponding function is positive and when it is negative.

Q1 Drag point *P* from Quadrant I to Quadrant II, and observe each distance at the transition. Some distances are 0, some are 1, and some increase without limit. Make a copy of the preceding table, and fill in the first "→" column with 0, 1, or ∞ to indicate the behavior of the distance corresponding to each function. (Three of the cells in this column are already filled in for you.)

Q2 When a distance is 1 during the transition, the sign of the function remains the same, but when the distance is either 0 or unbounded, the sign of the function changes. In column QII, enter the new sign for each function in Quadrant II.

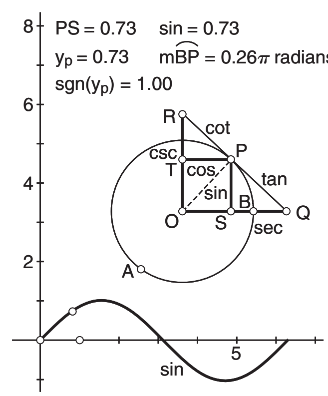
Q3 Observe the distances as you drag P from Quadrant II to Quadrant III. Fill in the next two columns of the table. Similarly, complete the rest of the table.

Signum (abbreviated sgn) is in the Calculator's Functions menu. This function returns a value based on the sign of the argument: 1 if it's positive, 0 if it's zero, or −1 if it's negative.

8. Measure the x- and y-coordinates of P. Use the Calculator to compute the values $\text{sgn}(x_P)$, $\text{sgn}(y_P)$, and $\text{sgn}(y_P/x_P)$.
9. Observe the behavior of these three calculations as you drag P into each of the four quadrants. For each distance measurement, there's one calculation that produces the desired sign for the corresponding function in your table. For instance, the calculation $\text{sgn}(x_P)$ produces the desired sign for the cos function.
10. Multiply distance PT by the $\text{sgn}(x_P)$ calculation, and label the result *cos.* This calculation gives the correct value of the cosine function for every position of P.
11. Similarly, multiply each of the other distances by the calculation that will produce the correct positive and negative values according to your table. Label each result based on the corresponding circular function.
12. Color each calculation to match the color of the corresponding segment. Hide the intermediate calculations, the measurements, and the coordinate axes.

To construct the arc, select the circle, point B, and point P, and choose **Construct | Arc On Circle.**

13. Construct the arc from B to P and measure its angle. If the measurement is in degrees, choose **Edit | Preferences** to set Angle Units to radians.
14. Construct a new point in empty space away from the unit circle, and choose **Graph | Define Origin** to define a new coordinate system. Hide the grid.
15. On the new coordinate system, plot the point $(m\widehat{BP}, \text{sin})$. Drag P to observe the behavior of the plotted point.



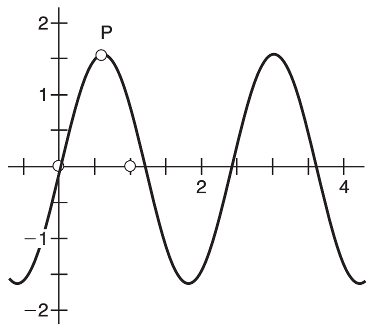
16. Construct the locus of the plotted point as P moves around the circle. Label this locus *sin,* and color it to match the corresponding segment and calculation.
17. Similarly, plot points and construct loci to match the other five segments.
18. For each circular function, create a hide/show button to hide or show all of its features (the segment, the calculated value, and the locus). Create an animation button to animate P around the circle. Use these buttons to present your work.

Sine Challenge

In this activity you will create a sine function whose amplitude and period are controlled by an independent point.

SKETCH AND INVESTIGATE

1. In a new sketch, choose **Edit | Preferences,** and set Angle Units to radians.
2. Construct parameters *a* and *b*. Use **Edit | Properties | Parameter** to set the keyboard adjustment for each parameter to 0.05.
3. Graph the function $y = a \cdot \sin(b \cdot x)$. Hide the grid.
4. Construct point *P* in a blank area of Quadrant I. Measure its *x*- and *y*-coordinates.
5. Use the + and − keys with parameters *a* and *b* to center the first crest of the sine graph at point *P*.



6. Select *a*, *b*, x_P, and y_P, and choose **Graph | Tabulate** to create a table. Double-click the table to make the first row permanent.
7. Move *P* to a new position in Quadrant I, and again adjust *a* and *b*. Double-click the table when the first crest is centered at *P*. Collect several more rows of data, making sure each time that the crest of the graph is centered at *P*.

Q1 As you look at the data in the table, what connection do you see between the value of *a* and the other values in the table?

Drag *P* all around, not just in Quadrant I.

8. Edit the function to eliminate parameter *a* and make the amplitude of the graph always match the position of *P*. Drag *P* to check your result.

To make a custom tool, press the custom tool icon at the bottom of the toolbox and choose **Create Custom Tool** from the menu that appears.

9. Determine how the value of *b* relates to other measurements, and edit the function to eliminate *b* while making sure that *P* is always at the first crest. Once you're satisfied, delete the table and parameters *a* and *b*.
10. Select point *P*, the function, and the graph and make a custom tool. Use the tool several times to make several easily controlled sine graphs.

EXPLORE MORE

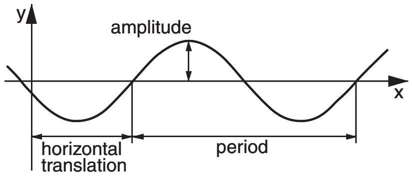
On a new page of the sketch, use your tool to create three sine functions. Define and graph a new function that is the sum of the three existing functions. Drag the points to see how the sum of the three functions relates to the three original functions.

Sums of Sinusoidal Functions

When you add two sinusoidal functions, the sum function has very interesting behavior that depends on the amplitude, period, and phase of the functions being added. You can use such functions to model many physical motions that exhibit periodic behavior. They are especially useful for analysis of waves.

SUMS OF FUNCTIONS

Begin by graphing several sine functions and examining the effects on the graph as the function parameters are varied.



1. Open **Trig Coords.gsp.**
2. Choose **Graph | Plot New Function** to plot the function $f(x) = \sin(x)$.

This is an elementary sine curve. The amplitude is 1, the period is 2π, and the horizontal translation is 0. (Horizontal translation is often called *phase.*)

To edit a function, double-click it with the **Arrow** tool.

Q1 What are the amplitude, period, and horizontal translation of each function below? Make a table of your predictions, and check them by editing $f(x)$.

$$f(x) = \sin(3x) \qquad f(x) = 4\sin(x)$$

$$f(x) = \sin\left(x - \frac{\pi}{2}\right) \qquad f(x) = 4\sin\left[3\left(x - \frac{\pi}{2}\right)\right]$$

Q2 Consider a sine curve in this general form: $f(x) = a \cdot \sin[b(x - c)]$. What are the amplitude, period, and horizontal translation?

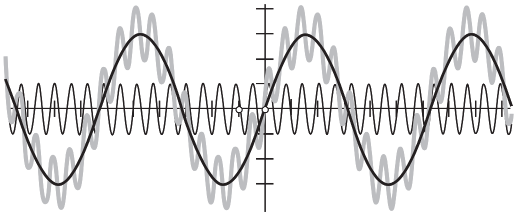
The principle of *wave superposition* states that where two or more waves come together, the resulting wave is simply the sum of the components. You can observe this relationship in many natural phenomena, including fluid waves, sound, and light.

Use color to match each function with its graph. Select the function definition and its graph, and choose **Display | Color** to set a new color.

Choose **Display | Line Width** to make the plot of $h(x)$ thick.

Investigate by plotting several curves on the same grid and finding their sum.

3. Edit the existing function definition to be $f(x) = 3\sin(x)$. Plot a new function, $g(x) = \sin(10x)$, and make it a different color from the previous graph. Plot another new function, $h(x) = f(x) + g(x)$. Make the plot of $h(x)$ thick, and use yet another color.

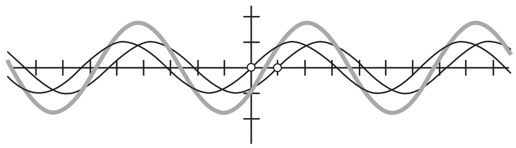


To understand the graph of $h(x)$, you can imagine that one of the sine curves acts as an axis for the other. Function h appears to be crawling along the length of f. Next you will see what happens when you combine two graphs that are more similar in appearance.

To create a hide/show button, select the objects and then choose **Edit | Action Buttons | Hide/Show.**

4. Create separate hide/show buttons for each of the three graphs. The graphs are easier to examine when you view them selectively.
5. Edit the function definitions for f and g:

$$f(x) = \sin(x) \qquad g(x) = \sin\left(x - \frac{\pi}{2}\right)$$

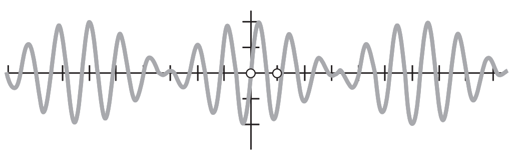


Q3 What are the period and horizontal translation for h? How do these compare with the characteristics of f and g?

6. Set these function definitions for f and g:

$$f(x) = \sin(6x) \qquad g(x) = \sin(5x)$$

This image shows only $h(x)$.



Q4 The graph of h shows clusters of waves called *wave packets,* which create an effect called *beats.* In sound waves, this is the pulsing effect that you hear when two notes have slightly different pitch. What is the length of each wave packet? The amplitude is variable. What is its greatest value?

Q5 The following table lists several different definitions for the functions in the form $f(x) = \sin(ax)$, $g(x) = \sin(bx)$. Try each combination on the sketch. In each case, record $a - b$ and the length of the wave packet in the graph of $h(x)$. What can you infer about the relationship between $a - b$ and the packet length?

$f(x) = \sin(ax)$	$g(x) = \sin(bx)$	$a - b$	packet length
$\sin(6x)$	$\sin(5x)$		
$\sin(10x)$	$\sin(9x)$		
$\sin(10x)$	$\sin(9.5x)$		
$\sin(10x)$	$\sin(8x)$		
$\sin(16x)$	$\sin(14x)$		

To create a parameter, choose **Graph | New Parameter** and type the desired parameter label into the dialog box.

To create an animation button, select the object to animate and choose **Edit | Action Buttons | Animation.** Set the desired direction, speed, and domain before you click OK.

When you study wave behavior, the waves are usually moving. You can simulate this motion in your model.

7. Create a new parameter t and set its initial value to 0. Create an animation button for t, and set the animation to continuously increase on the domain from 0 to 1000.

8. Set these function definitions for f and g:

$$f(x) = \sin[6(x - t)] \qquad g(x) = \sin[5(x - t)]$$

Q6 Press the animation button. As t increases, how do the speed and direction of h compare with the speed and direction of its components, f and g?

9. Now examine the sum of two graphs that have slightly different periods and are moving in opposite directions. Enter these function definitions:

$$f(x) = \sin[6(x - t)] \qquad g(x) = \sin[5(x + t)]$$

Q7 How do these definitions make the waves of h move in opposite directions? What changes took place in the speed and direction of travel of the wave packets? Do they have the same length?

Q8 Switch back and forth between the function definitions in steps 8 and 9, and describe and compare the wave behavior. Which definitions result in simpler behavior of h? Why does one pair of definitions result in more complex behavior?

Q9 What do you think the results will be if the two functions have the same period and amplitude, but move in opposite directions? Write down your conjecture about the amplitude, period, and motion of h.

10. Change the definitions so that the functions have the same period, but opposite directions:

$$f(x) = \sin(x - t) \qquad g(x) = \sin(x + t)$$

Q10 Animate parameter t. Was your prediction correct? Describe the motion of the graph of h. This effect is called a standing wave.

Save the sketch you created. You can use it in the activity Products of Sinusoidal Functions.

PHYSICS CONNECTIONS

Q11 Describe something in the physical world that exhibits the standing wave behavior seen in step 10.

Q12 In step 9, the wave packets move much faster than the component parts, f and g. Some people have proposed that this effect might enable radio communications that travel faster than light. Can you identify the flaw in this reasoning?

EXPLORE MORE

Experiment by combining more than two sine functions. For instance, graph the sum of the following functions:

$$f(x) = \sin(x) \qquad g(x) = \frac{1}{3}\sin(3x) \qquad h(x) = \frac{1}{5}\sin(5x)$$

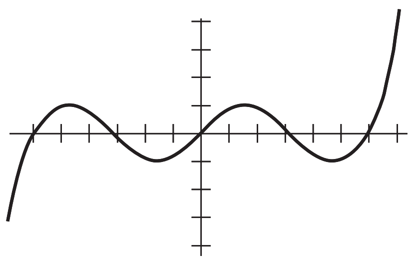
Add several more terms to this sum, following the same pattern, and describe the resulting function. If you added many more terms, what do you think the graph would look like?

Taylor Series

You can compute the value of a polynomial function directly and easily for any particular value of *x* using multiplication and addition. But values of other functions, such as the sine function, are much more difficult to compute.

You can learn how to derive the Taylor series when you study calculus.

In this activity you'll approximate the sine function using a series called a *Taylor series* and observe the behavior of the partial sums when the series is evaluated to various depths. The Taylor series approximation for sin(x) is



$$f(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$$

SKETCH AND INVESTIGATE

1. In a new sketch, create a square grid, construct point *A* on the *x*-axis, and measure the point's *x*-value. Label the *x*-value *x*.

2. Create four parameters to use in iterating the series. Label them *i, num, den,* and *sum.*

Q1 Parameter *i* represents the odd integers, following the sequence 1, 3, 5, ... What rule can you apply to one element of this sequence to calculate the next?

Q2 Parameter *num* represents the numerator, taking on values x, $-x^3$, x^5, $-x^7$, and so forth. What is the rule to calculate a value of this sequence from the previous value?

Q3 Parameter *den* represents the denominator, taking on values 1!, 3!, 5!, and so forth. What's the rule to calculate the next value of this sequence? (Express your answer in terms of the previous values of *den* and *i.*)

Q4 Parameter *sum* represents the sum of all the terms from the first term through the *i*th term. What value should you use as the initial value of the sum, before adding the very first term? What's the rule to calculate one sum from the previous sum?

Parameters must be independent values in order to be iterated, so you can't set the initial values of the parameter that depends on *x* until after you construct the iteration.

3. All but one of these parameters have constant initial values that you can assign now. (The initial value for the other isn't constant, but depends on the value of *x.*) Assign appropriate initial values to the parameters that don't depend on *x.* Assign an initial value to the other parameter as though the value of *x* were 2.

Q5 What initial values did you assign to the parameters?

Make sure the calculated values are what you expect.

4. For each of the four parameters, use the rule you described above to calculate the next value of the quantity it represents. (Your calculations should involve only the values of the four parameters and the value of *x*.)

5. Plot the point (*x*, *sum*). The iterated image of this plotted point will allow you to see the graph of each successive expansion of the Taylor series.

Select all four parameters, and choose **Transform | Iterate.** Then match each parameter to its next value.

6. Iterate each of the parameters to its calculated next value.

7. The parameter *num* doesn't yet have a correct initial value, because the initial value depends on *x*. Select *num*, choose **Edit | Edit Parameter,** and calculate the initial value so that it depends correctly on *x*.

8. Drag point *A* left and right on the *x*-axis, observing the values in the table and the positions of the plotted points.

The last row of the table should contain $n = 1$.

9. Select the iterated image of the plotted point and press the **–** key on the keyboard twice to set the depth of iteration to 1.

The *terminal point* is the very last image of the iterated point, based on the current depth of iteration.

10. With the iterated image of the plotted point still selected, choose **Transform | Terminal Point.** Then construct the locus of the terminal point as *A* moves along the axis.

Q6 What is the shape of the locus? Which terms contribute to this shape?

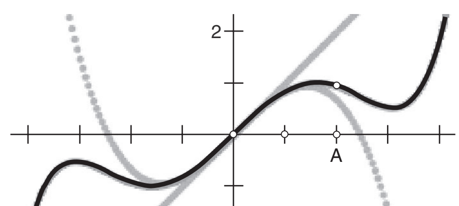
To increase the depth, select either iterated image (the table or the image of the plotted point), and press the **+** key on the keyboard.

11. Set the depth of iteration to 2.

Q7 How does this change the shape of the locus? Which terms contribute now?

12. Increase the depth to 3. Turn on tracing for both the iterated image of the plotted point and the locus. Animate point *A*, and observe the behavior of the point images.

Q8 What shapes do the iterated point images trace? Sketch their shapes and explain the role of each trace based on the terms of the series.



n	i + 2	−num · x^2	den · (i + 1) · (i + 2)	sum + $\frac{\text{nu}}{\text{de}}$
0	3.00	−8.00	6.00	2.00
1	5.00	32.00	120.00	0.67
2	7.00	−128.00	5040.00	0.93
3	9.00	512.00	362880.00	0.91

You may have to move the origin and change the domain of the locus to see two full periods.

13. While point *A* is moving, increase the depth until the locus accurately approximates the sine curve for at least two periods.

Q9 How many terms are required to give a reasonable approximation for the first period of the sine function? For the first two periods?

EXPLORE MORE

The Taylor series for the cosine function is as follows:

$$f(x) = \frac{x^0}{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$$

You can change the initial values of your existing iteration to calculate this series. Decide which parameters to change, and calculate and plot the modified series.

Tracing the Slope Function

With a linear function like $f(x) = 2x - 3$, you can use any two points to calculate Δy and Δx and calculate the slope of the function. But what happens when you try to find the slope of a function that isn't linear?

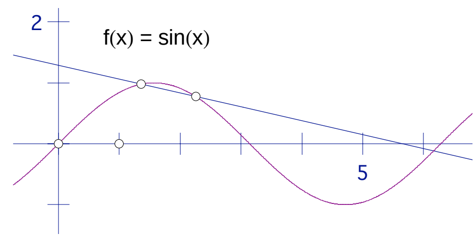
MEASURE THE SLOPE

In the Calculator, choose **sine** from the Calculator's value menu and then click the **x** key.

1. In a new sketch, graph $f(x) = \sin x$ by choosing **Display | Plot New Function.**
2. When you click OK in the Calculator, Sketchpad may ask whether the coordinate system should use radians rather than degrees. Answer Yes to this question.
3. Press and hold the **Segment** tool icon until a pop-out menu appears. Choose the **Line** tool (with arrowheads on both ends).

Be sure you click directly on the graph.

4. Construct a secant line to the function plot by clicking the **Line** tool in two places fairly near each other on the graph. Switch back to the **Selection Arrow** tool.



The Motion Controller appears when you choose this command. If you've never used it before, experiment with its controls.

5. With the secant line selected, choose **Display | Animate Line.**

The secant line begins moving along the graph to the right.

Q1 Describe the motion of the secant line in detail. What changes as it moves from left to right? What happens when the points reach the edge of the screen? What else do you notice?

If you have stopped the animation, start it again by selecting the secant line and choosing **Display | Animate Line.**

6. With the secant line still selected and moving, choose **Measure | Slope.**

Q2 What does this measurement represent? What do you observe about it? Be specific about the value of the measurement. What are the largest and smallest values you observe? When do they occur? What else do you notice?

PLOT AND TRACE

Q3 You can graph the slope measurement to help you understand its behavior. For this graph, the slope will be the dependent variable. You will also need an independent variable. What does the slope depend on? What should be the independent variable? (Give as many reasonable answers as you can think of.)

7. With the secant line still moving, select point A by pressing the Target area of the Motion Controller and choosing Point A from the menu that appears.
8. Measure the abscissa (x-coordinate) of Point A.

> To plot the point, select in order the independent variable (x_A) and the dependent variable (*Slope AB*), and choose **Graph | Plot As (x, y)**.

9. Plot the point determined by the two changing measurements.

Q4 Observe the behavior of the plotted point that appears. How does it move horizontally? How does it move vertically?

10. With the plotted point selected, choose **Display | Trace Plotted Point.**

Q5 As the plotted point continues moving, it leaves a trace behind. Describe this trace. Watch if for several trips back and forth, and be sure to describe everything that you observe. In particular, describe what happens when points *A* and *B* reach either edge of the screen.

Q6 Does the trace look like it's the graph of a function? Why or why not? (Is there any test you know that you can use to answer this question?)

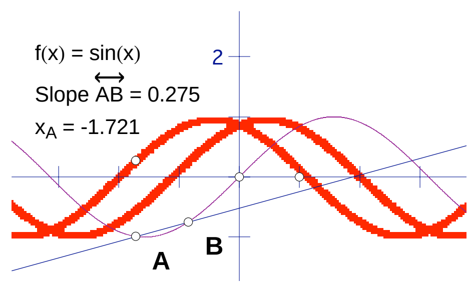
Q7 What do you think would happen if points *A* and *B* were closer together while being animated? What if they were farther apart?

> The Stop button is the one shown with a square.

11. Click the Motion Controller's Stop button to stop the animation.

12. Select points *A* and *B*. Make an Animation button by choosing **Edit | Action Buttons | Animation** and accepting the default settings. Press the button several times to start and stop the animation.

13. With the animation stopped, move *A* and *B* farther apart, restart the animation, and observe the trace.



Q8 How does the trace change when the points are farther apart? Be sure to look at the period, the minimum and maximum values, and any other differences you see.

14. Stop the animation, move *A* and *B* closer together, and restart the animation.

Q9 How does this change affect the trace?

15. Stop the animation, move *A* halfway to *B*, and then move *B* until it's directly on top of *A*.

Q10 What happens? Why?

16. Move one of the points so they're slightly separated and restart the animation.

Q11 Describe the trace that you see now. What are the maximum and minimum values now? What other differences are there compared to previous traces?

Q12 Does the traced point now trace a function? If so, justify your answer. If not, explain why not.

Tracing the Slope Function

Objective: Students construct a secant line to a function, observe the behavior of the line's slope, and move the defining points of the line closer together to simulate a tangent line. This activity develops an intuitive idea of limits and of the derivative as the limit of the slope of a function.

Student Audience: Precalculus, Calculus

Sketchpad Level: Intermediate.

Activity Time: 30 minutes. Allow plenty of time for discussion.

Setting: Paired/Individual Activity or Whole-Class Presentation (no sketch needed)

MEASURE THE SLOPE

Q1 Observations will vary. The line's intersections with the function plot move across the screen, first from left to right and then back from right to left. Each intersection point bounces off the edge of the screen. As the line moves, it tilts up and down, tilting up by about the same amount as it tilts down. Near the peaks and valleys of the sine function, it's horizontal; it's steepest when it's near the roots of the sine function plot.

Q2 This measurement represents the slope of the line. The largest value is between 0.9 and 1.0, and occurs near the spot where the sine function crosses the *x*-axis from below to above. The smallest value is between –0.9 and –1.0, near the spot where the sine function crosses the *x*-axis from top to bottom.

PLOT AND TRACE

Q3 The slope varies as the secant line's control points move left and right, so it depends on the positions of these control points. The independent variable should be the positions of the control points.

In fact, there can be only a single independent variable, so in the next step we choose one of the two points arbitrarily.

Q4 The plotted point moves horizontally in lock-step with point *A*. Its vertical position corresponds to the value of the slope of the secant line.

Q5 The trace goes up and down as the secant line points up and down. When *A* reaches the edge of the screen, it bounces back, and so does the plotted point. The shape of the trace changes slightly while the two points are going in opposite directions.

Q6 The trace does not appear to be the graph of a function, because there are two different values for each value of *x*. The trace fails the vertical-line test.

Q7 Answers will vary. Students may predict that the two branches of the trace will also be closer together, and that they will be farther apart when the points are farther apart. The important thing is that they make a conjecture first, before trying the experiment.

Q8 When the two points are farther apart, the two braches of the trace become farther apart, and the maximum and minimum values of the slope are reduced. (In other words, the trace flattens out somewhat.) The period of the trace remains the same.

Q9 When the points are closer, the two branches of the trace are closer, and the extrema become closer and closer to +1 and –1. The period remains the same.

Q10 When the two points are exactly on top of each other, the secant line, the slope measurement, and the plotted point all disappear, because it takes two distinct points to define a line.

Q11 When the two points are nearly on top of each other, the trace shows only a single branch, looking like the trace of a function. The maximum and minimum values are now nearly +1 and –1. The period remains the same.

Q12 The plotted point now appears to trace out a function. Students may observe that it's still not a function, because if we looked closely enough we would still see two branches, very close together.

Use Q10 and Q12 to spur a class discussion. Ask students how they would have to adjust the points to get the traces to coincide exactly. (For this to happen, points *A* and *B* would also have to coincide exactly.) Ask them what this would mean for the secant line. (A line requires two points, and would not be well-defined if it had only a single defining point. With *A* and *B* as one single point, the slope and the trace also would not exist.) This is a good place to develop an intuitive understanding of limits—students could put the two points close enough to achieve any desired closeness of the branches of the trace, as long as they don't make them coincide precisely.

WHOLE-CLASS PRESENTATION

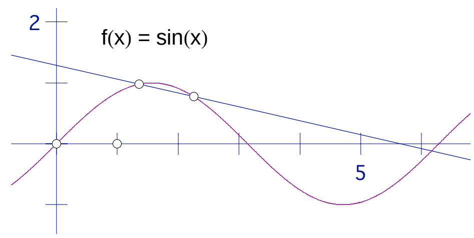
This activity makes a wonderful presentation. See the Presenter Notes for suggestions.

Use this presentation to spur student observation, conjectures, and discussion concerning the concept of limit applied to a secant line and its slope.

1. In a new sketch, graph $f(x) = \sin x$. If Sketchpad asks whether the coordinate system should use radians, answer Yes.

Be sure you click directly on the graph.

2. Construct a secant line on the graph.



3. With the secant line selected, choose **Display | Animate Line.**

Leave the animation running during the questions and discussion.

Q1 Ask students to describe the motion of the secant line in detail. What changes as it moves from left to right? What happens when the points reach the edge of the screen? What else do they notice?

If you have stopped the animation, start it again by selecting the secant line and choosing **Display | Animate Line.**

4. With the secant line still selected and moving, choose **Measure | Slope**.

Q2 Ask students what this measurement represents. What do they observe about it? Ask them to be specific about the value of the measurement. What are the largest and smallest values they observe? When do they occur? What else do they notice?

Q3 You can graph the slope measurement to better understand its behavior. For this graph, the slope will be the dependent variable. Ask students what the independent variable should be. Encourage discussion; settle on the *x* value of *A*.

5. With the secant line still moving, select *A* by pressing the Target area of the Motion Controller and choosing Point *A*. Measure its *x*-coordinate, and plot the point determined by x_A and the slope.

With the plotted point selected, choose **Display | Trace Plotted Point**.

Q4 Ask students to observe and describe this point's behavior. Turn on tracing for the point, and ask students to observe and describe the trace that it leaves behind.

Q5 Ask students if the trace appears to be the graph of a function. How can they tell?

Q6 Ask students what would happen if points *A* and *B* were closer together while being animated? What if they were farther apart?

The Stop button is the one shown with a square. Use **Display | Erase Traces** to remove the traces from the screen.

6. Stop the animation and make an Animation button that animates both *A* and *B*. Use the button to start and stop the animation and observe the results when the points are farther apart, and when they are closer together.

7. Move *A* and *B* as close as you can get them and animate again.

Q7 Have students describe this trace. What are the maximum and minimum values now? What other differences are there compared to previous traces? Ask students whether the traced point now traces a function. Ask them to justify their answers.

8. Finish with a class discussion about why you need to get *A* and *B* closer and closer. What would happen if *A* and *B* were exactly on top of each other?