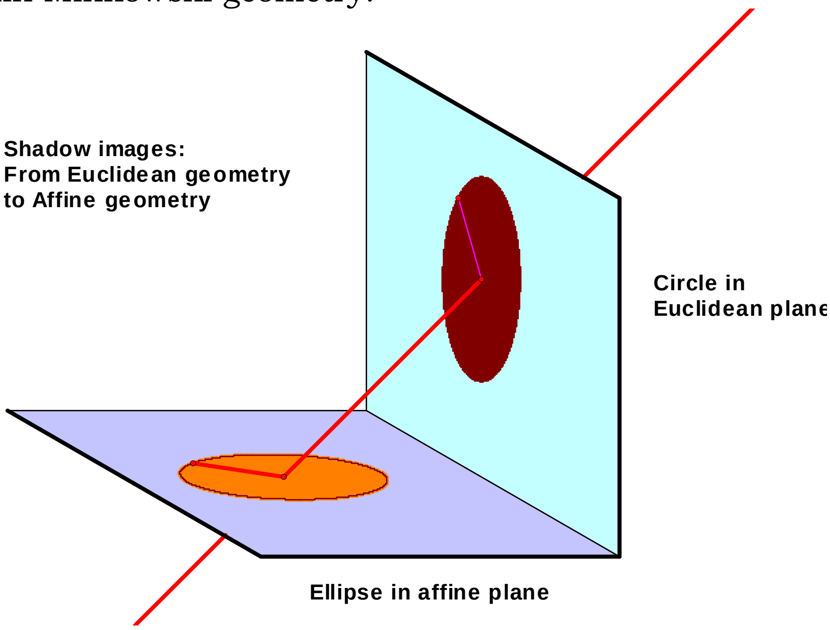
Workshop 2 in Minkowski Geometry: Conic sections
Shadowing Euclidean and Minkowski Geometry

In the second part of the workshop we will investigate conic sections in various ways. First we will compare traditional constructions of conic sections in the two geometries to develop our intuition about in particular Minkowski Geometry. Especially we will focus upon the construction of central conic sections using circles or equiangular hyperbolas as starting points thus introducing foci and directrices etc. This will be fairly standard although a few small surprises will pop up within Minkowski geometry.

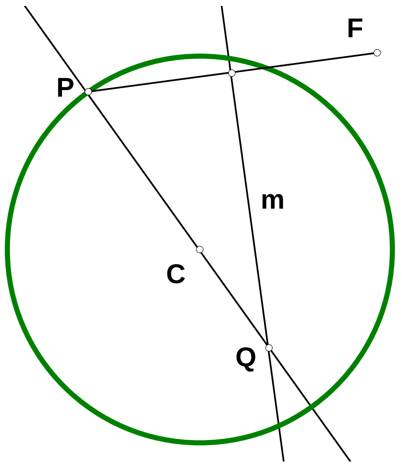


Next we will extend our point of view to affine geometry. Central conics in affine geometry may be considered affine projections, i.e. shadow images, from Euclidean geometry or Minkowski geometry. In this way affine theorems about central conics may be considered generalizations of corresponding theorems in Euclidean and Minkowski geometry. The most general theorems incorporate both ellipses and hyperbolas depending upon the particular position of various base points. Thus we will need some insight into the **classification of central conics**: Under what conditions do we get ellipses and under what conditions do we get hyperbolas? There will also be intermediary cases, which will show up as degenerate central conics, i.e. either a pair of two parallel lines or a double line. The special cases of circle theorems in Euclidean Geometry or equiangular hyperbolas in Minkowski geometry will help us to throw light upon these classifications.

We will consider in particular two such theorems, the first being a classical theorem known as the 9-point conic section associated with a triangle (which generalizes the famous 9-point circle or Feuerbachs circle). The other case is not so well known but was drawn to my attention by an inspiring lecture on the classification of affine conics '*Om kjeglesnitts bestemmelse ved hjelp av rette linjer*' (i.e. *'On the determination of conic sections using straight lines*) by *Signe Holm Knudtzon* and *Johan Aarnes* at the Norwegian conference '*Ski og matematikk* 2004'. Although my focus differs somewhat from theirs the entire example grew out of their presentation.

Example 1: Folding conic sections

Euclidean geometry:



Construct a circle with centre *C* and a *free point F not* on the periphery. Construct as well a *free point P* on the periphery of the circle.

Construct the perpendicular bisector *m* of the segment *FP* as well as the intersection point *Q* with the line through *C* and *P* (i.e. the extended radius!)

Construct the *locus M* of *Q* driven by the circle point *P*. What kind of locus do we construct in this way? How does it depend upon the free point *F*. What is the role of the perpendicular bisector *m*?

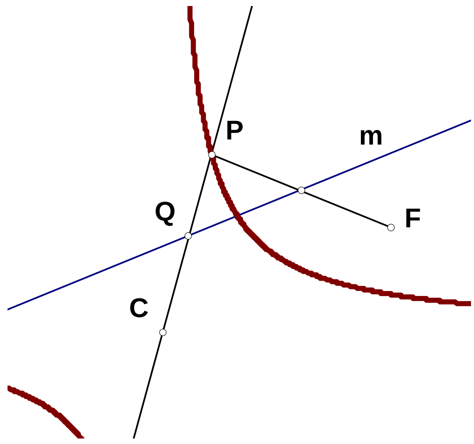
Measure the Euclidean distances *CQ* and *FQ*. How can we characterize *M* using these distances?

Construct the extended line through *C* and *F* as well as the perpendicular bisector of the segment *CF*. What are the special roles of these two lines?

Construct the line through *F* perpendicular to the segment *FQ*. Construct the intersection point *D* with the perpendicular bisector *m*. Construct the locus *l* of *D* driven by *P*. What is the significance of *l*?

Measure the Euclidean distances *FQ* and *Fl*. How can we characterize the locus *l* through these distances?

Minkowski geometry:



Construct a hyperbola with centre *C* and a *free point F not* on the periphery. Construct as well a *free point P* on the periphery of the hyperbola.

Construct the perpendicular bisector *m* (in Minkowski sense!) of the segment *FP* as well as the intersection point *Q* with the line through *C* and *P* (i.e. the extended radius!)

Construct the *locus M* of *Q* driven by the hyperbola point *P*. What kind of locus do we construct in this way? How does it depend upon the free point *F*. What is the role of the perpendicular bisector *m*?

Measure the Minkowski distances *CQ* and *FQ*. How can we characterize *M* using these distances?

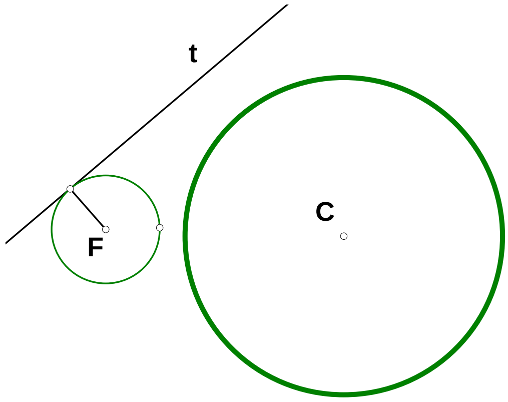
Construct the extended line through *C* and *F* as well as the perpendicular bisector of the segment *CF* (in Minkowski sense!*)*. What are the special roles of these two lines?

Construct the line through *F* perpendicular to the segment *FQ*. Construct the intersection point *D* with the perpendicular bisector *m*. Construct the locus *l* of *D* driven by *P*. What is the significance of *l*?

Measure the Minkowski distances *FQ* and *Fl*. How can we characterize the locus *l* through these distances?

Example 2: Reciprocating conic sections

Euclidean geometry:



Construct a 'unit' circle with centre *C* and another reference circle with centre *F*. Construct as well a *free point P* on the periphery of the reference circle. Construct the tangent *t* at the point *P*.

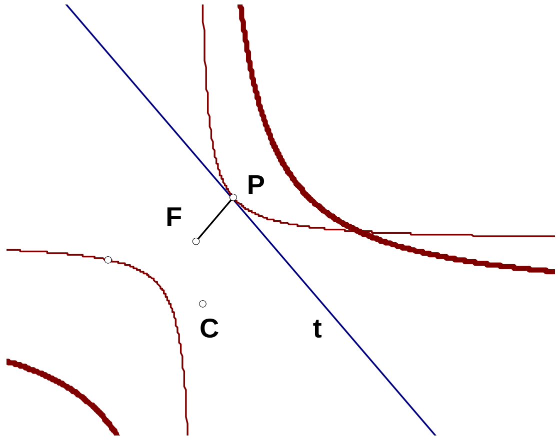
We are now going to *reciprocate* this tangent *t* in the unit circle. We will do two constructions for the reciprocating point *Q*: One for the case, where the tangent *t* intersects the unit circle, one where the tangent *t* falls outside the unit circle.

Case I: Construct the intersection points *A* and *B* between the tangent *t* and the unit circle. Construct the tangents at *A* and *B* to the unit circle. They intersect at the reciprocating point *Q*.

Case II: Construct the perpendicular to the tangent *t* through the centre *C* intersecting each other at C_0. Construct the circle with the diameter CC_0. It intersects the unit circle in *A* and *B*. The chord AB intersects the diameter CC_0 in the reciprocating point *Q*.

Construct the locus of the reciprocating point *Q* driven by the point *P*. What do you observe about the locus?

Minkowski geometry:



Construct a 'unit' hyperbola with centre *C* and another reference hyperbola with centre *F*. Construct as well a *free point P* on the periphery of the reference hyperbola. Construct the tangent *t* at the point *P*.

We are going to *reciprocate* this tangent *t* in the unit hyperbola. We will do two constructions for the reciprocating point *Q*: One for the case, where the tangent *t* intersects the unit hyperbola, one where the tangent *t* falls outside the hyperbola.

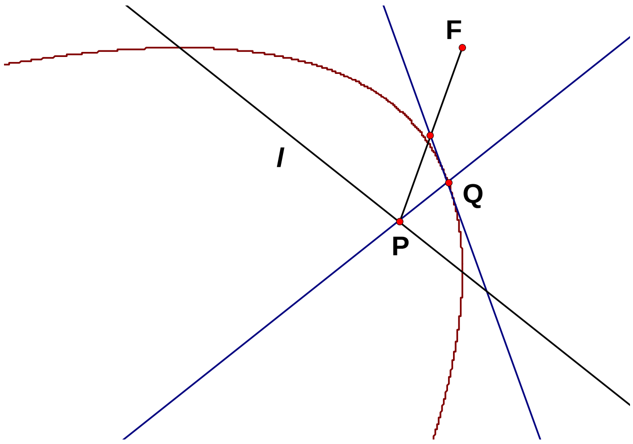
Case I: Construct the intersection points *A* and *B* between the tangent *t* and the unit hyperbola. Construct the tangents at *A* and *B* to the unit hyperbola. They intersect at the reciprocating point *Q*.

Case II: Construct the hyperbolic perpendicular to the tangent *t* through the centre *C* intersecting each other at C_0. Construct the hyperbola with the diameter CC_0, intersects the unit hyperbola in *A* and *B*. The chord AB intersects the diameter CC_0 in the reciprocating point *Q*.

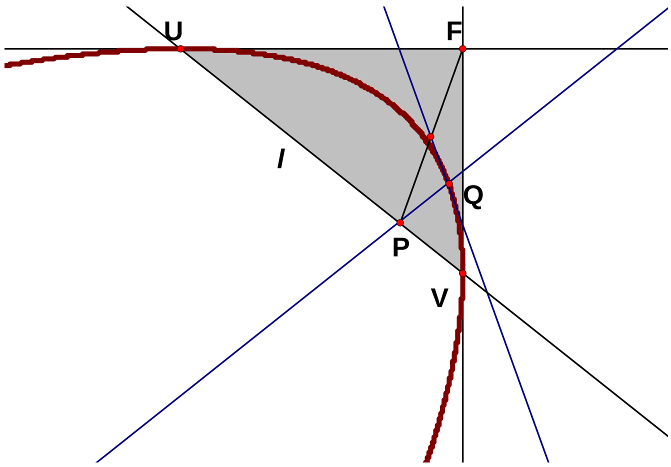
Construct the locus of the reciprocating point *Q* driven by the point *P*. What do you observe about the locus?

A note on conic sections in Minkowski Geometry

In the previous exercises we have generated conic sections using standard procedures. Especially the folding procedure is intimately related to the foci of the conic sections: It allows for the construction of the conic section on the basis of two focal points and a point on the periphery. This construction can then be recorded as a macro. This allows us to investigate conic sections from a Minkowski point of view. The basic difference between Euclidean Geometry and Minkowski Geometry is the existence of null lines in Minkowski geometry. They give rise to null points for the general conic, i.e. points where the tangents are null lines. And these null points actually generate the basic properties of the conic sections from a Minkowski point of view. Let us illustrate this with the familiar construction of a parabola generated by a focus point *F* and a directrix *l*. If we select a free point *P* on the directrix we can construct a point *Q* on the parabola as the intersection between the perpendicular bisector of *FP* and the perpendicular to the directrix through *P*. The parabola is thus constructed as the locus of *Q* driven by *P*:

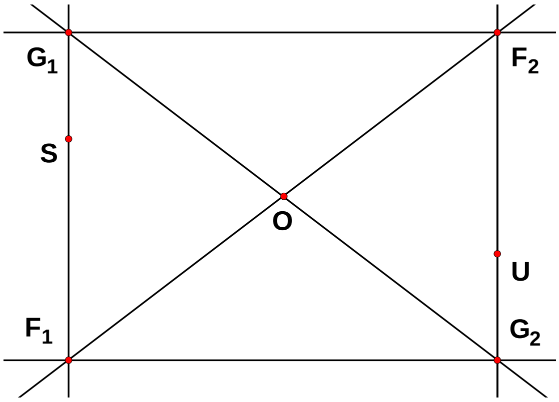


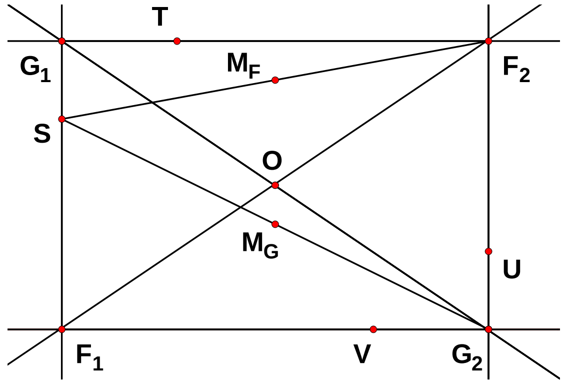
The interesting points are the intersection between the null lines through F and the directrix. They simply intersect at the null points of the parabola. This shows that the null points of the parabola generate both the directrix and the focus in a very simple manner. Notice that you can also verify the basic distance property of the parabola: $|QF| = |Ql|$ using the measurement tool for Minkowski distances.



Similar phenomena occur for the ellipse and the hyperbola as indicated in the following exercises.

Example 3: The ellipse as a conic section in Minkowski Geometry





Select two free points F_1 and F_2 and construct the *null lines* trough F_1 and F_2. They intersect at the points G_1 and G_2. Construct the diagonals through F_1 and F_2 as well as G_1 and G_2. They intersect at the centre point of the enclosing null lines O.

What can you say about the diagonals F_1F_2 and G_1G_2?

Select a free point S on the null segment F_1G_1. Reflect the point S in the centre point O to obtain the point U.

Construct the lines through S and U parallel with the diagonals F_1F_2 and G_1G_2. They intersect the remaining sides F_2G_1 and F_1G_2 in the points T and V.

Investigate the quadrilateral $STUV$. What kind of quadrilateral have you constructed?

Construct the rectangular hyperbolas with centre F_2 and G_2 passing through S. Where do they intersect the null lines?

Construct the rectangular hyperbolas with centre F_1 and G_1 passing through U. Where do they intersect the null lines?

Construct the segments SF_2 and SG_2 as well as their midpoints M_F and M_G.

(to be continued!)

Construct the hyperbolas with centre F_2 and peripheral points M_F respectively centre G_2 and peripheral point M_G. They intersect the diagonals F_1F_2 and G_1G_2 in four points known as the *vertex points:* A_1, A_2 on the diagonal F_1F_2 and $B1$, $B2$ on the diagonal G_1G_2.

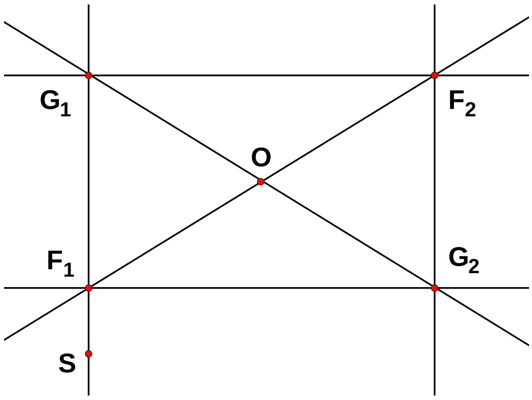
Construct a free point P on any of the four hyperbolas passing through S, T, U and V. Locate the centre of the hyperbola as well as the opposite corner. Complete the construction of the ellipse using the folding construction from the previous exercise. What do you observe?

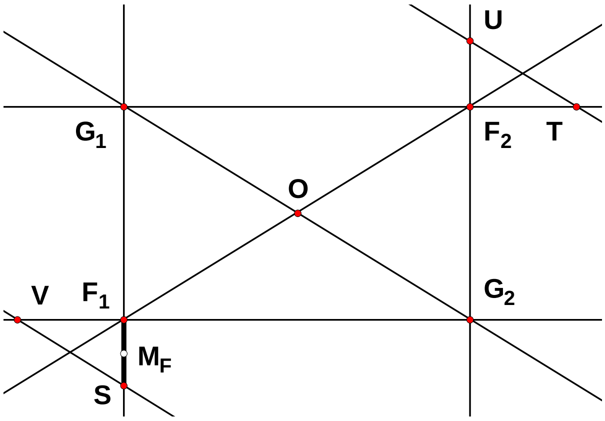
Construct a free point E on the ellipse. Measure the Minkowski distances from E to the four corner points F_1, F_2, G_1 and G_2. Argue that all four corner points serve as focal points for the ellipse.

Construct the tangent associated with E as well as the normal. Construct the focal lines F_1E, F_2E, G_1E and G_2E. Measure the angles between the focal lines and either the tangent or the normal associated with E. What do you observe?

Specialise finally to the case where S is the midpoint of F_1G_1. This can e.g. be done by splitting S from the line F_1G_1 and then combining S with the midpoint. What can you say about this special ellipse?

Example 4: The hyperbola as a conic section in Minkowski Geometry





Select two free points F_1 and F_2 and construct the *null lines* trough F_1 and F_2. They intersect at the points G_1 and G_2. Construct the diagonals through F_1 and F_2 as well as G_1 and G_2. They intersect at the centre point of the enclosing null lines O.

What can you say about the diagonals F_1F_2 and G_1G_2?

Select a free point S on the null line F_1G_1 *outside* the segment F_1G_1. Reflect the point S in the centre point O to obtain the point U.

Construct the lines through S and U parallel with the diagonal G_1G_2. They intersect the remaining null lines F_2G_1 and F_1G_2 in the points T and V.

Investigate the quadrilateral $STUV$. What kind of quadrilateral have you constructed?

Construct the rectangular hyperbola with centre F_2 passing through S. Construct the rectangular hyperbola with centre F_1 passing through U. Where do these hyperbolas intersect the null lines?

Construct the segments SF_1 as well as the midpoint M_F. Construct the hyperbola with centre O and peripheral points M_F. It intersects the diagonal F_1F_2 in two points known as the *vertex points* A_1, A_2.

Construct the tangents at the vertices as lines parallel to the diagonal G_1G_2. Construct the hyperbola with centre O passing through the focus F_1. Construct the four intersection points with the tangents. Construct the lines connecting O to these intersection points. These are going to be the *asymptotes* of the hyperbola.

Construct a free point P on the rectangular hyperbola passing through T and U with F_1 as the centre. Complete the construction of the hyperbola using the folding construction from the previous exercise with F_2 as the other focus. What do you observe?

Construct a free point E on the hyperbola. Measure the Minkowski distances from E to the four corner points F_1, F_2, G_1 and G_2. Argue that all four corner points serve as focal points for the hyperbola.

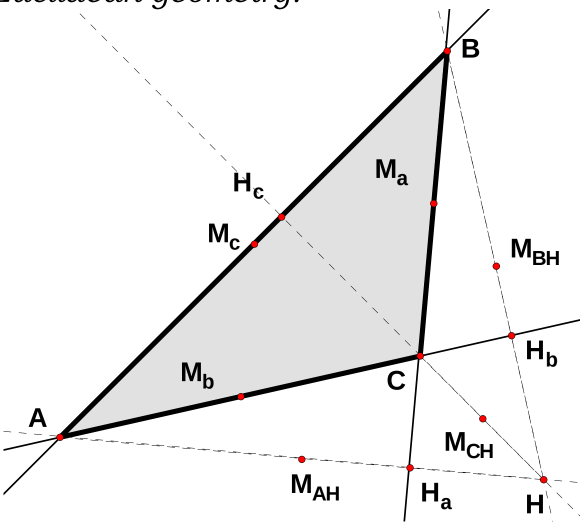
Construct the tangent associated with E as well as the normal. To do this you must reverse the folding construction, since the perpendicular bisector is the tangent!

Construct the focal lines F_1E, F_2E, G_1E and G_2E. Measure the angles between the focal lines and either the tangent or the normal associated with E. What do you observe?

(to be continued!)

Example 5: The nine-point conic

Euclidean geometry:



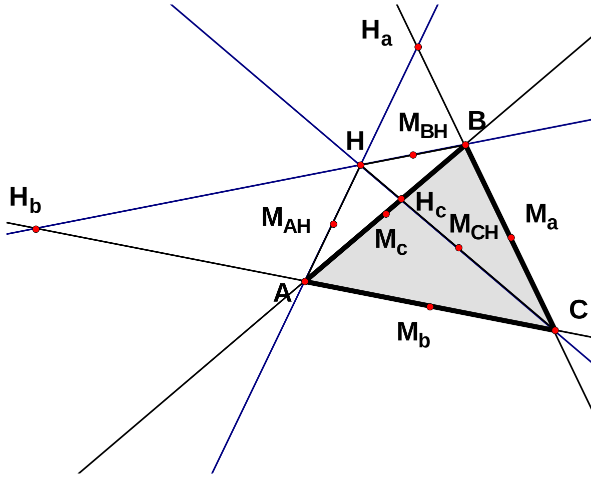
Construct a triangle *ABC* with extended lines as sides. Construct the midpoints M_a, M_b and M_c of the segments *BC*, *CA* and *AB*. Construct the altitudes from *A*, *B* and *C*. They intersect the opposite sides in H_a, H_b and H_c. The altitudes intersect in the othocentre *H*. Construct the midpoints M_{AH}, M_{BH} and M_{CH} of the segments *AH*, *BH* and *CH*. Thus you have now constructed nine points associated with the triangle. Construct the general conic passing through five of these points. What do you observe?

Under an affine projection the right angles are lost. So we now replace the orthocentre *H* with a free point *S* and the three altitudes with the three concurrent lines from the vertices A, B and C passing through *S*. Complete the previous construction!

Move *S* around! What kind of conics can you generate this way? Under what circumstances do you generate ellipses? hyperbolas? What can you say about the transition from the ellipses to the hyperbolas? Can you construct parabolas this way?

Argue that when you generate an ellipse this way, it may be considered an affine projection of a triangle with its nine-point circle!

Minkowski geometry:



Construct a triangle *ABC* with extended lines as sides. Construct the midpoints M_a, M_b and M_c of the segments *BC*, *CA* and *AB*. Construct the Minkowski altitudes from *A*, *B* and *C*. They intersect at the orthocentre H and the opposite sides in H_a, H_b and H_c. Construct the midpoints M_{AH}, M_{BH} and M_{CH} of the segments *AH*, *BH* and *CH*. You have now constructed nine points associated with the triangle. Construct the general conic passing through five of these points. What do you observe?

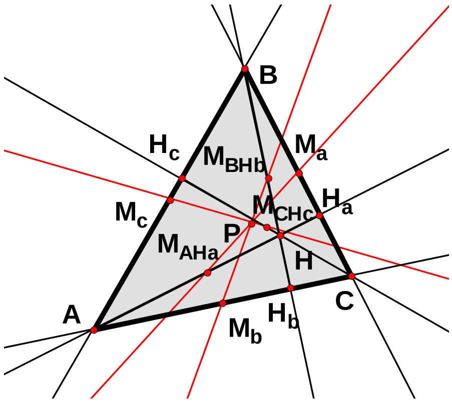
Under an affine projection the right angles are lost. So we now replace the orthocentre *H* with a free point *S* and the three altitudes with the three concurrent lines from the vertices A, B and C passing through *S*. Complete the previous construction!

Move *S* around! What kind of conics can you generate this way? Under what circumstances do you generate ellipses? hyperbolas? What can you say about the transition from the ellipses to the hyperbolas? Can you construct parabolas in this way?

Argue that when you generate a hyperbola this way, it may be considered an affine projection of a triangle with its nine-point hyperbola!

Example 6: The parallel generated conic

Euclidean geometry:



Construct a triangle *ABC* with extended lines as sides. Construct the midpoints M_a, M_b and M_c of the segments *BC*, *CA* and *AB*. Construct the altitudes from *A*, *B* and *C*. They intersect the opposite sides in H_a, H_b and H_c. Construct the midpoints of the altitudes M_{AHa}, M_{BHb} and M_{CHc}. Construct the three lines connecting the midpoints of the sides with the midpoints of the altitudes. They intersect at the *circular point P*.

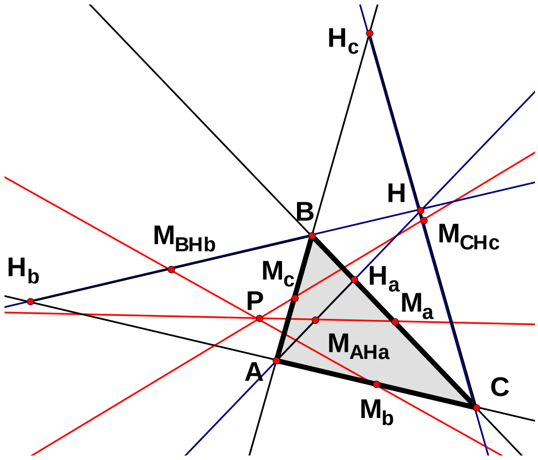
Draw the parallels to the sides *a*, *b* and *c* through the circular point *P*. They intersect the sides in six points P_{ab},…, P_{cb}. Construct the general conic passing through five of these points. What do you observe?

Under an affine projection right angles are lost. So we now replace the circular point *P* with a free point *T*. Complete the previous construction!

Move *S* around! What kind of conics can you generate this way? Under what circumstances do you generate ellipses? hyperbolas? What can you say about the transition from the ellipses to the hyperbolas? Can you construct parabolas this way?

Argue that when you generate an ellipse this way, it may be considered an affine projection of a triangle with its circle generated by parallels through with the circular point.

Minkowski geometry:



Construct a triangle *ABC* with extended sides. Construct the midpoints M_a, M_b and M_c of the segments *BC*, *CA* and *AB*. Construct the Minkowski altitudes. They intersect the opposite sides in H_a, H_b and H_c. Construct the midpoints of the altitudes M_{AHa}, M_{BHb} and M_{CHc}. Construct the three lines connecting the midpoints of the sides with the midpoints of the altitudes. They intersect at the *hyperbolic point P*.

Draw the parallels to the sides *a*, *b* and *c* through the hyperbolic point *P*. They intersect the sides in six points P_{ab},…, P_{cb}. Construct the general conic passing through five of these points. What do you observe?

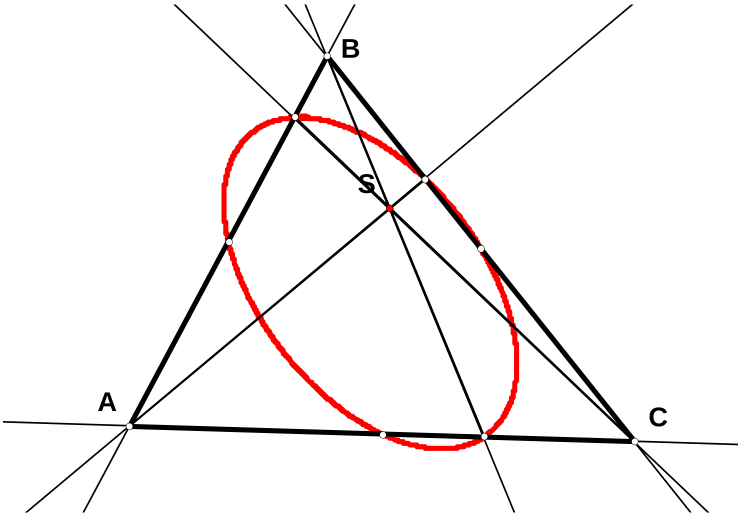
Under an affine projection right angles are lost. We now replace the hyperbolic point *P* with a free point *T*. Complete the previous construction!

Move *S* around! What kind of conics can you generate this way? Under what circumstances do you generate ellipses? hyperbolas? What can you say about the transition from the ellipses to the hyperbolas? Can you construct parabolas this way?

Argue that when you generate a hyperbola this way, it may be considered an affine projection of a triangle with its hyperbola generated by parallels through the hyperbolic point.

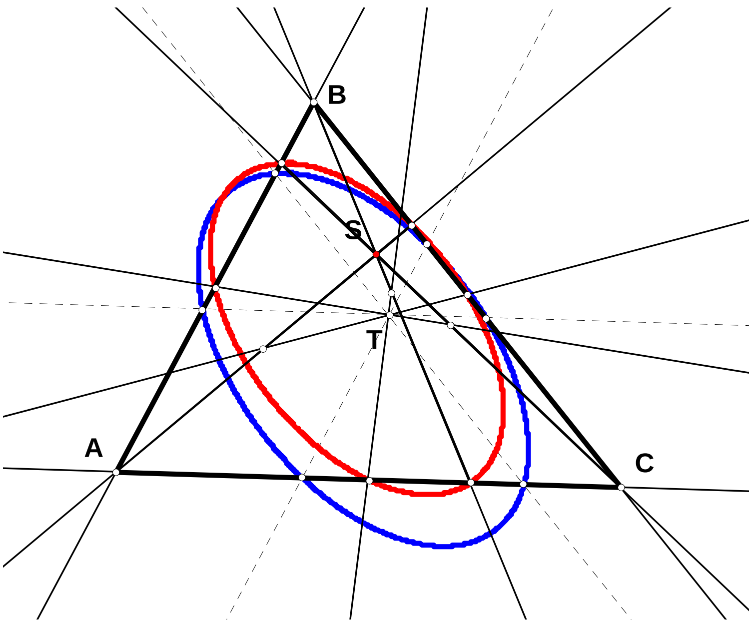
Bridging the gap: Connecting the nine-point conic and the parallel generated conic

The two constructions are some what similar, so this suggests that you can actually bridge the nine-point conic and the parallel generated conic. The idea is to extend the transition from the orthocentre to the circular point. The starting point is a triangle *ABC* with extended sides and midpoints M_a, M_b and M_c:

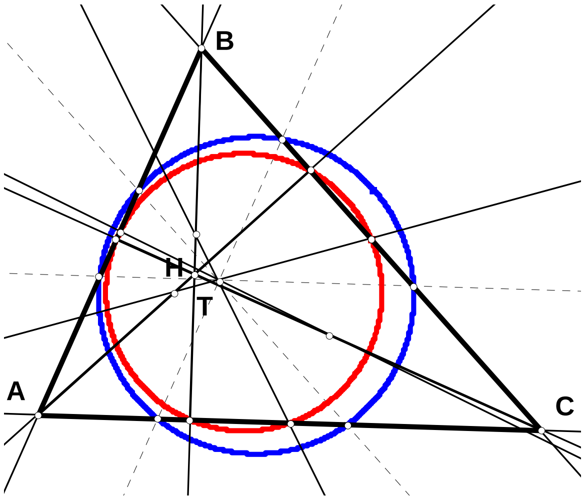


If *S* is a free point we can draw the concurrent lines through *S* and the three vertices *A*, *B* and *C*. They intersect the extended sides in S_a, S_b and S_c. The six points M_a, M_b, M_c, S_a, S_b and S_c generate the nine-point (red) conic.

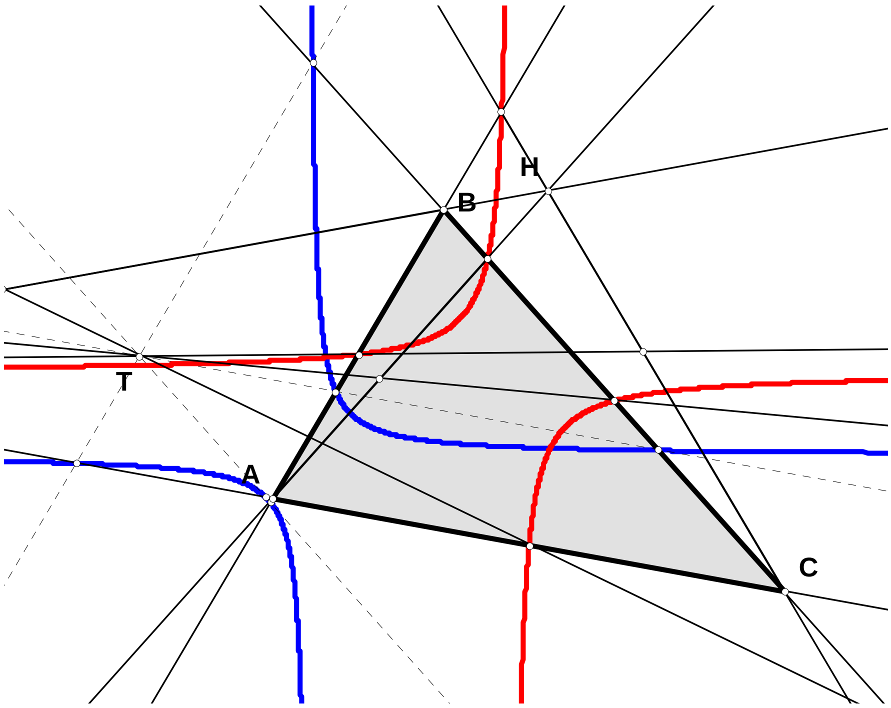
But we can also construct the midpoints of the three concurrent lines AS_a, BS_b and CS_c, i.e. M_{ASa}, M_{BSb} and M_{CSc}. Constructing the three lines connecting the midpoints of the triangle sides with the midpoints of the concurrent lines, we get a new set of concurrent lines intersecting in *T*.



The parallels through *T* intersect the extended sides of the triangle in six points P_{ab}, …, P_{ca} that generate the parallel generated (blue) conic.



If *S* happens to coincide with the Euclidean orthocentre of the triangle then *T* corresponds to the circular point, and thus both conics are circles. If *S* happens to coincide with the Minkowski orthocentre of the triangle then *T* corresponds to the hyperbolic point and both conics are rectangular hyperbolas.

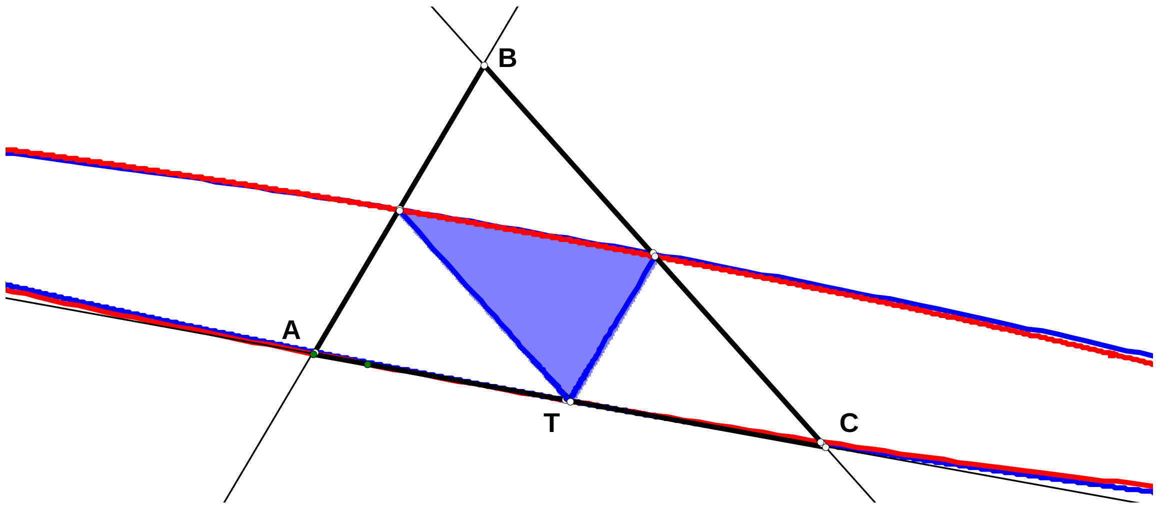


But this correspondence continues: The conics generated by S and T are similar in general. Thus we can learn about the classification of conics in one case by studying the classification in the other case.

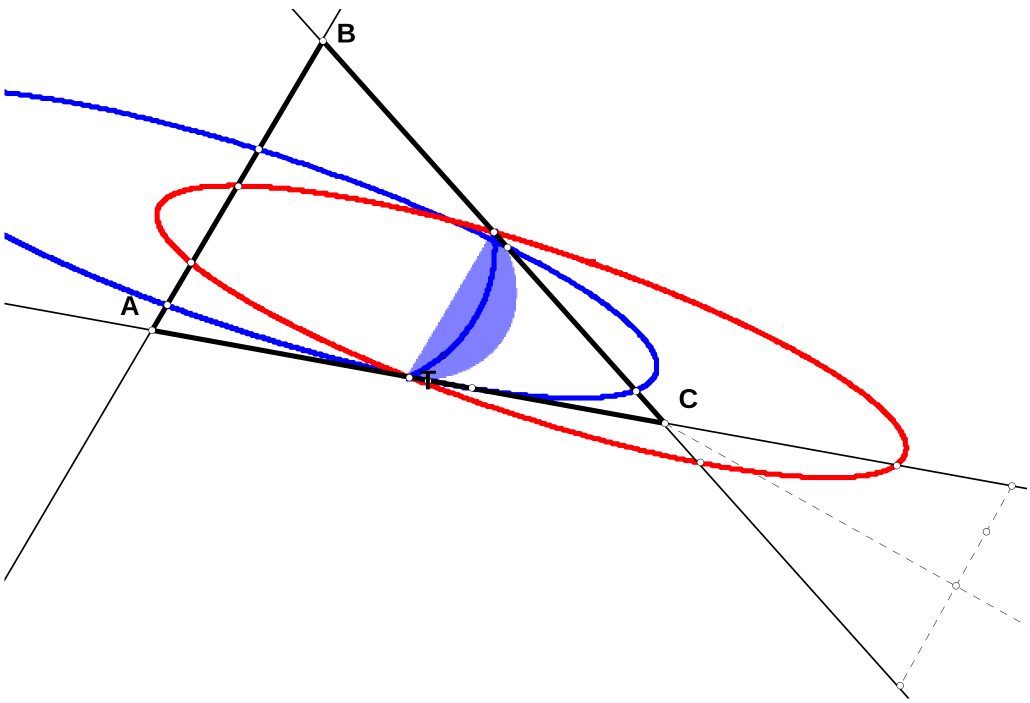
Now the nine-point conic is easy to classify: If S falls inside the triangle or inside one of the external angles then the resulting conic is an ellipse. If S falls in one of the regions bounding a side, then it is a hyperbola. The transition happens when we pass one of the (extended) sides and the transition conic is a pair of parallel lines except when S coincide with a vertex of the triangle. Here the conic degenerates completely!

This classification is thus translated into a similar classification of the parallel generated conics using the non-linear map from *S* to *T* described above. We must thus find out, what happens to T, when S belongs to various regions:
The simplest case to consider is when S falls on one of the extended sides (but not a vertex!). Then T coincides with the midpoint of this side, so the mapping is highly degenerate when we pass the sides!

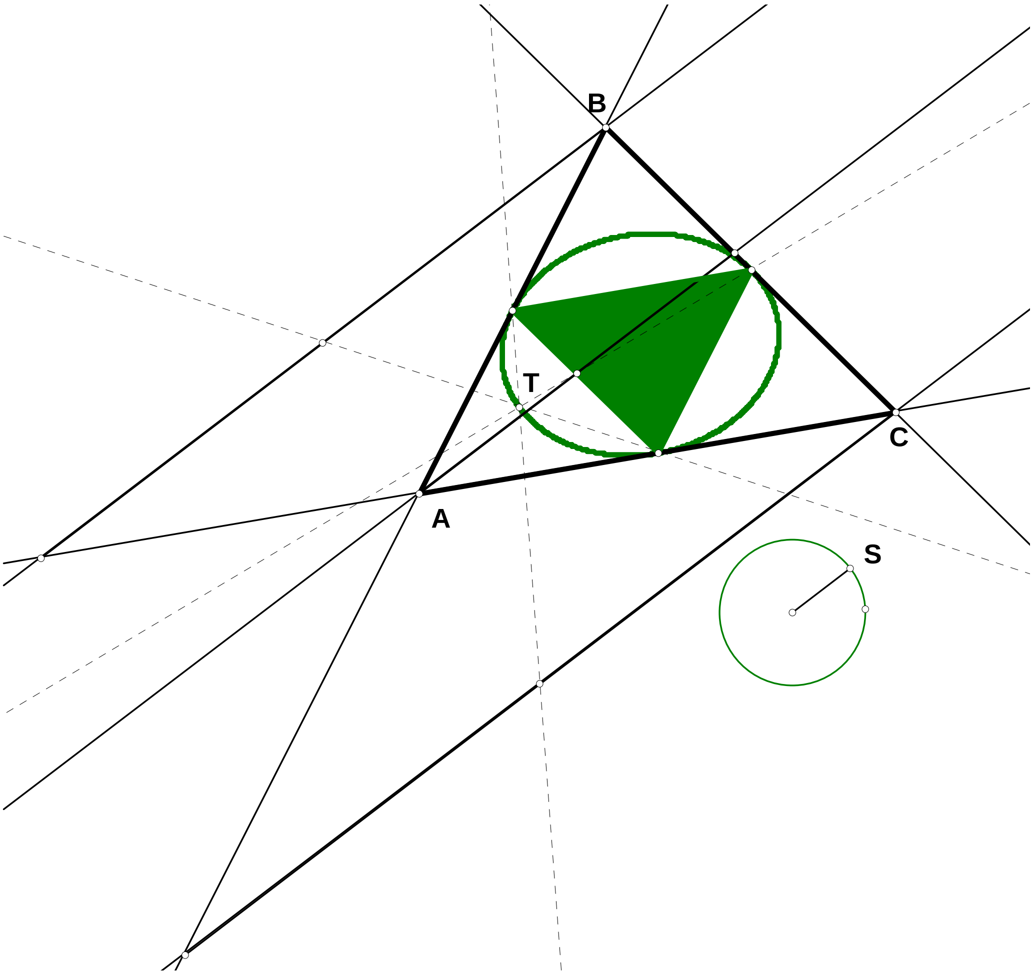
Next we can let S sweep through the interior of the triangle and trace T (to make life a little easier we let S sweep a parallel to one of the sides and construct the locus of T. Tracing this locus as the parallel sweeps through the triangle then gives an easy overview of the range of T). The result is surprisingly simple: The interior of ABC is mapped onto the interior of the midpoint triangle:



Next we look at one of the exterior angles, e.g. the exterior angle at *C*. This is somewhat more difficult to experiment with as it is unbounded. You might think that you could then cover the reaming part of the interior angle at *C*, but it turns out not to be the case:



The nine-point conic has a transition curve at infinity and this is actually mapped onto the midpoint ellipse of the triangle *ABC*, i.e. the unique ellipse passing through the mid points with the sides of the triangle as its sides. You can check this by letting *S* be an infinitely far point and thus let it be represented by its direction:



It is also easy to reverse the map, i.e. to construct *S* from *T*: Construct the midpoint triangle i.e. the triangle generated by the midpoints of the original triangle. Extend the sides. Connect the point *T* with the midpoint of the original triangle. Construct the intersection points with the extended sides of the midpoint triangle. Connect the intersection points with the vertices of the original triangle. This produces three concurrent lines intersecting at *S*.

In this way you can make a detailed study of the classification of the parallel generated conics associated with a triangle.

Remark: As shown by Johan Aarnes and Signe Holm Knudtzon the construction of the parallel generated conic is actually more general. You can use the point *T* as a similarity centre and consider the intersections between the original triangle *ABC* and the similarity transformed triangle. This creates six intersection points that generate a conic. The type of the conic is independent of the similarity factor. The above discussion corresponds to the similarity factor 0!