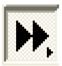
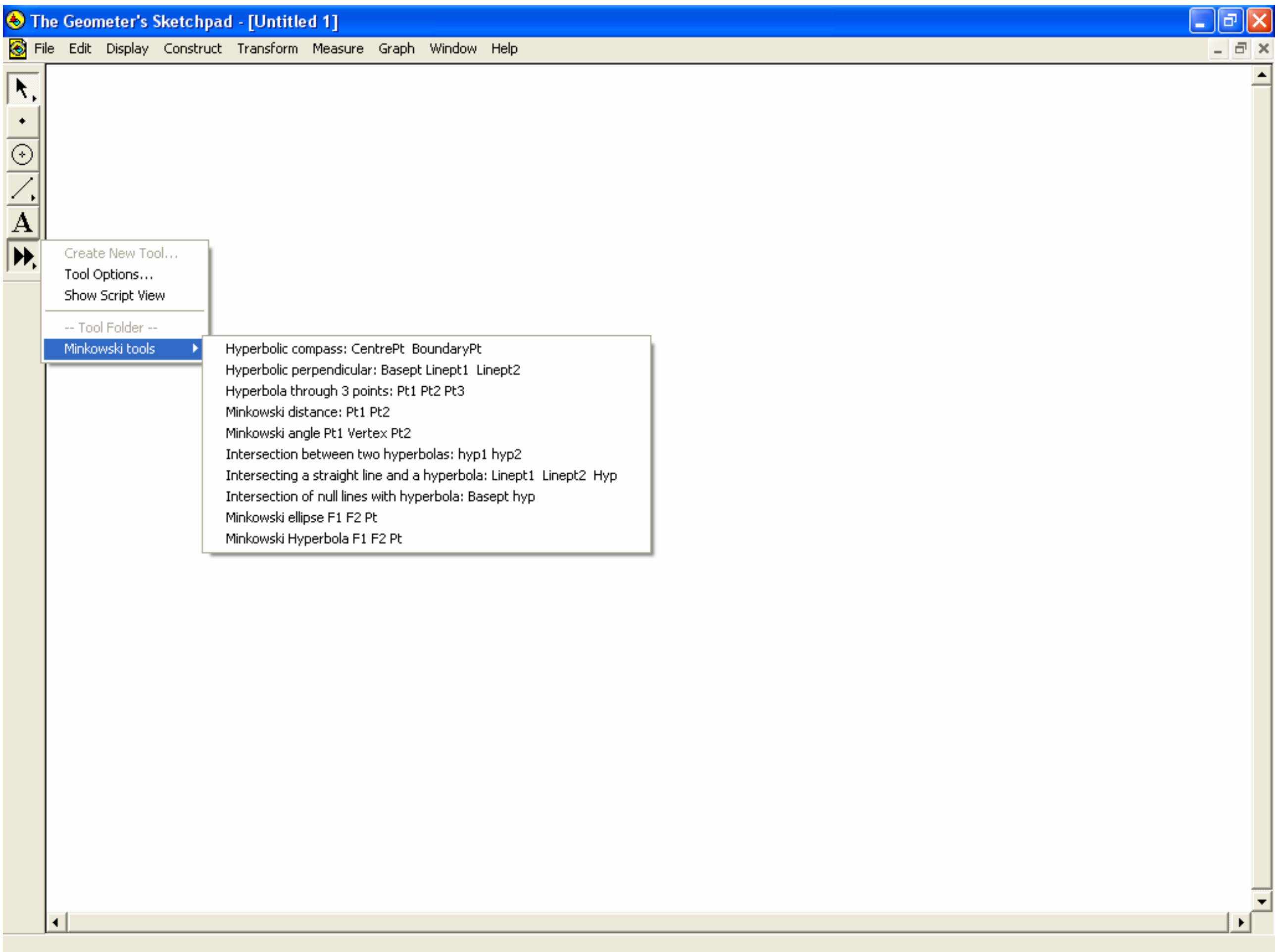
Workshop 1 on Minkowski Geometry using SketchPad

Preliminary remarks: To perform geometrical constructions in Minkowski geometry, you need some hyperbolic drawing requisites such as a **hyperbolic compass** for drawing rectangular hyperbolas and a **hyperbolic perpendicular template** for the construction of hyperbolic perpendiculars.

A **hyperbolic compass** is a compass which allows you to draw the rectangular hyperbola with the given centre and the given peripheral point. The axes as always in Minkowski Geometry are assumed to be horizontal and vertical.

A **hyperbolic perpendicular template** is an instrument used for drawing a hyperbolic perpendicular to a line from a point which may or not be on the line. In Minkowski Geometry we typically use an instrument based upon three points: The first two points are used to determine the direction of the straight object: a line, a ray or a segment. The last base point is the point through which the perpendicular must pass.

It is convenient to use electronic devises to simulate the above-mentioned instruments. This can done using **macros** in **Sketchpad**. Such macros can either be opened during a particular session, or they can be registered as general purpose tools in **SketchPad** by collecting them in a special document – **Minkowski Tools** – stored in a special folder carrying the name **Tool Folder**. On the main screen in **Sketchpad** we are now offered a new icon  on the toolbar. If we press down the mouse a list of general-purpose tools available is shown:



In this case we thus have the basic construction tools **the hyperbolic compass** and **the hyperbolic perpendicular** at our disposal. Before you start working on the exercises you may benefit from experimenting a little with these tools to get a feeling for how they work:

The hyperbolic compass (using a centre point and a peripheral point): You must first select the centre. Subsequently the rectangular hyperbola pops up on the screen ready to be dropped at a given peripheral point.

The hyperbolic perpendicular (using a prescribed point and a base line). To construct a hyperbolic perpendicular you must first select a base point through which the perpendicular must pass. Next you select two points on the base line (the straight object). When the first line point has been selected the perpendicular pops up on the screen ready to be dropped. To construct a **hyperbolic bisector** you must therefore first construct the midpoint of the segment, and then drop a hyperbolic perpendicular selecting the end points of the segment.

Other hyperbolic tools
Notice that you also have a hyperbola through three points at your disposal as a construction tool as well as a number of intersection tools involving hyperbolas since SketchPad does not support intersections with loci. Finally you have various measurement tools for measuring the distance between two points and the angle spanned by three points. They replace the corresponding Euclidean tools.

Tools common with Euclidean Geometry
What is the relevance of the other tools in SketchPad? Some become superfluous being limited strictly to Euclidean geometry: This is the case e.g. for all tools dealing with circles or angles. Others are common for the Euclidean geometry and the Minkowski geometry. Especially noteworthy are the following **construction tools**:

- **Point: Point on object, Midpoint, Intersection**
- **Line: Segment, Ray, Parallel line**
- **Polygon: Interior**
- **Curve: Locus**

Certain **transformation tools** are also common:

- **Translate …**
- **Rotate – but only with 180° i.e. a point symmetry!**
- **Dilate …**

But general rotations as well as reflections are restricted to the Euclidean Geometry since they involve Euclidean angles and must be constructed on the fly when needed in Minkowski Geometry!

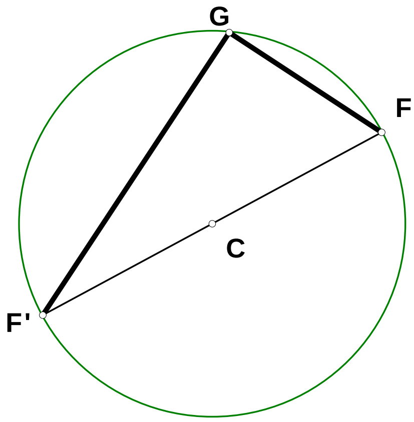
Finally certain **measurements tools** are also common:

- **Area**
- **Ratio**
- **Slope**

The area is common because the symmetry transformations of the Minkowski Geometry all have determinants 1 and thus they all preserve Euclidean areas. It follows that the Minkowski area and the Euclidean area are the same (up to a common proportionality factor, which is conveniently taken to be 1.

Exercise 1: The theorem of Thales

Euclidean geometry:



Construct a circle and a *free point F* on the periphery.

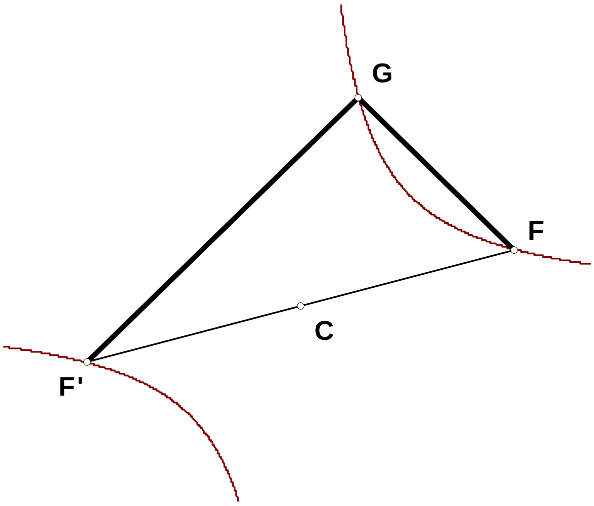
Using a reflection in the centre *C* we now obtain the *opposite point F'* on the periphery.

At this point we have thus succeeded in constructing a circle with a diameter *FF'*.

Select another *free point G* on the periphery.

Verify that the angle *FGF'* is a right angle using slopes.

Minkowski geometry:



Construct a rectangular hyperbola and a *free point F* on the periphery.

Using a reflection in the centre *C* we now obtain the *opposite point F'* on the periphery.

Remark: The reflection in the centre is technically implemented as a rotation by 180° around the centre. You might consider extending the radius *CF* until it intersects the hyperbola in the opposite point. You must then use the intersection tool for straight objects and hyperbolas.

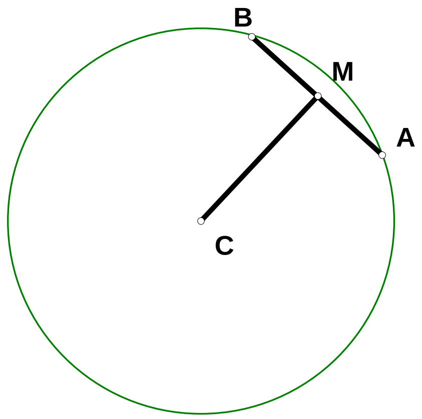
At this point we have thus succeeded in constructing a rectangular hyperbola with a diameter *FF'*.

Select another *free point G* on the periphery.

Verify that the angle *FGF'* is a hyperbolic right angle using slopes.

Exercise 2: The perpendicular bisector of a chord

Euclidean geometry:



Construct a circle with the centre *C* and two *free points A* and *B* on the periphery.

Construct the *chord AB*, i.e. the segment joining *A* and *B*.

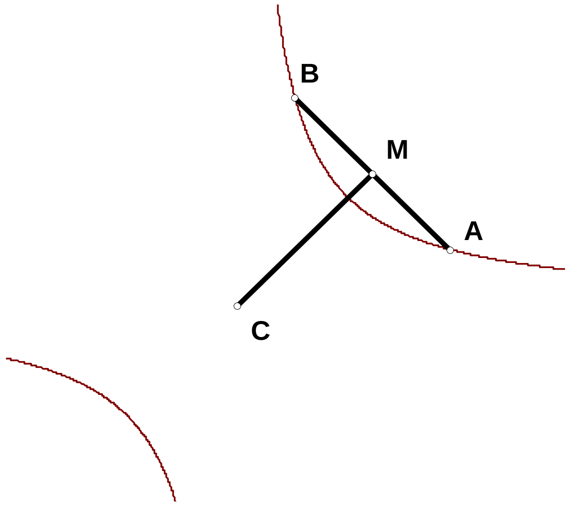
Construct the midpoint *M* for this chord. Construct the associated radius *CM*.

Verify that the angle between the chord *AB* and the radius *CM* is a right angle using slopes.

Conclusion: *The perpendicular bisector of a chord passes through the centre of the circle.*

Remark: What happens with this theorem, when the two points *A* and *B* coincide?

Minkowski geometry:



Construct a rectangular hyperbola with the centre *C* and two *free points A* and *B* on the periphery.

Construct the *chord AB*, i.e. the segment joining *A* and *B*.

Construct the midpoint *M* for this chord. Construct the associated radius *CM*.

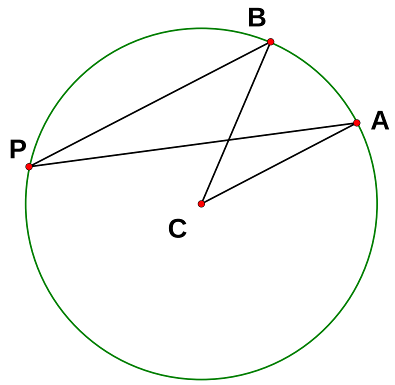
Verify that the angle between the chord *AB* and the radius *CM* is a hyperbolic right angle using slopes.

Conclusion: The perpendicular bisector of a chord passes through the centre of the rectangular hyperbola.

Remark: What happens with this theorem, when the two points *A* and *B* coincide?

Exercise 3: Angle games

Euclidean geometry:



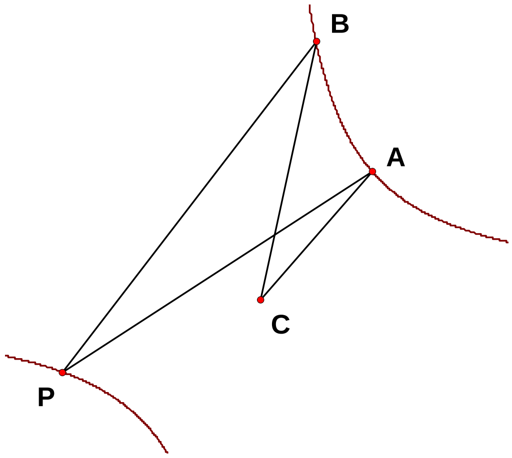
Construct a circle with the centre *C* and two *free points* *A* and *B* on the periphery. Select a third free point *P* on the periphery. Measure the peripheral angle *APB*. Move *P* around. What do you observe? Does it make any difference if *P* is inside or outside the arc *AB*?

Measure the centre angle *ACB*. What is the relation ship between the centre-angle and the peripheral angle?

Remove the circle!

Consider the segment *AB* and a free point *C* outside *AB*. How would you construct the locus of points from which *AB* is seen under the same angle as from *C*?

Minkowski geometry:



Construct a rectangular hyperbola with the centre *C* and two *free points* *A* and *B* on the periphery. Select a third free point *P* on the periphery. Measure the peripheral angle *APB* (using Minkowski tools!). Move *P* around. What do you observe? Try also to measure the slopes *PA* and *PB*. What is the relations ship between the two slopes?

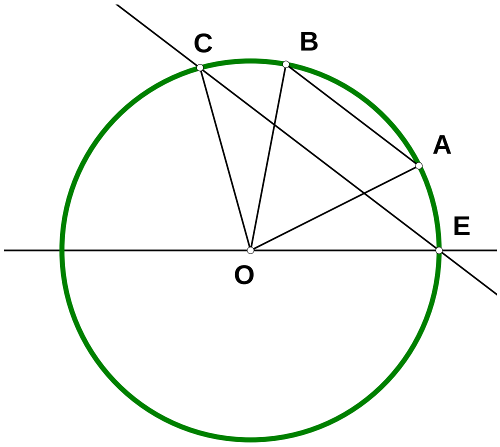
Measure the centre angle *ACB*. What is the relation ship between the centre-angle and the peripheral angle?

Remove the hyperbola!

Consider the segment *AB* and a free point *C* outside *AB*. How would you construct the locus of points from which *AB* is seen under the same angle as from *C*?

Exercise 4: Adding up angles

Euclidean geometry:



Construct a circle with the centre *O* and the unit point *E* on the periphery. Construct two *free points A* and *B* on the periphery as well as the chord *AB* joining the two points. Construct a parallel to the chord passing through the unit point *E*. The parallel intersects the circle in the third point *C*.

Measure the angles *A=EOA*, *B=EOB* and *C=EOC*. What is the relationship between *A*, *B* and *C*?

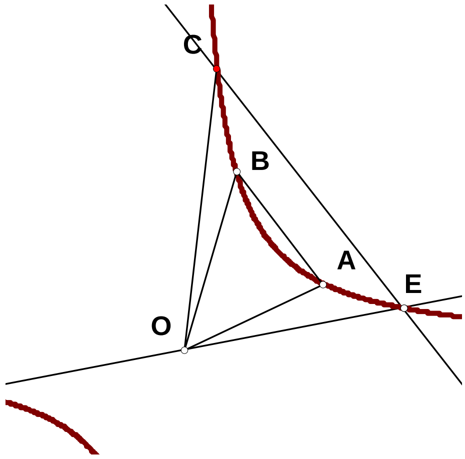
Measure the angle *BOC*. What do you observe?

** The rest of the exercise is slightly more complicated!*
We now modify the construction. Let *B* be the reflection of the unit point E in the radius segment *OA*. Construct *C* as before. Construct the parallels through *B* and *C* parallel with *OA*. What do you observe?

The parallel through *C* intersects the circle in the *D* and the centre line *OE* in *P*. Measure the angles *DOP* as well as *COD*. What do you observe?

Measure the distance *DP*. What do you observe?

Minkowski geometry:



Construct a hyperbola with the centre *O* and the unit point *E* on the periphery. Construct two *free points A* and *B* on the periphery as well as the chord *AB* joining the two points. Construct a parallel to the chord passing through the unit point *E*. The parallel intersects the circle in the third point *C*.

Measure the Minkowski angles *A=EOA*, *N=EOB* and *C=EOC*. What is the relationship between *A*, *B* and *C*?

Measure the angle *BOC*. What do you observe?

** The rest of the exercise is slightly more complicated!*
We now modify the construction. Let *B* be the hyperbolic reflection of the unit point E in the radius segment *OA*. Construct *C* as before. Construct the parallels through *B* and *C* parallel with *OA*. What do you observe?

The parallel through *C* intersects the circle in the *D* and the centre line *OE* in *P*. Measure the angles *DOP* as well as *COD*. What do you observe?

Measure the Minkowski distance *DP*. What do you observe?

Exercise 5: The perpendicular bisectors of a triangle

Euclidean geometry:

Construct a triangle *ABC.*

Construct the midpoints for the three legs.

Construct the perpendicular bisectors for the three legs.

What characteristic property of the perpendicular bisectors have you now successfully demonstrated?

Construct a circle passing through a vertex using the intersection point of the perpendicular bisectors as the centre.

What is the characteristic property of the circle thus constructed?

Remark: What happens when the intersection point of the perpendicular bisectors (the circum centre) falls on one of the legs of the triangle?

What kind of triangle possesses this property?

Minkowski geometry:

Construct a triangle *ABC.*

Construct the midpoints for the three legs.

Construct the hyperbolic perpendicular bisectors for the three legs.

What characteristic property of the perpendicular bisectors have you now successfully demonstrated?

Construct a rectangular hyperbola passing through a vertex using the intersection point of the perpendicular bisectors as the centre.

What is the characteristic property of the hyperbola thus constructed?

Remark: What happens when the intersection point of the perpendicular bisectors (the circum centre) falls on one of the legs of the triangle?

What kind of triangle possesses this property?

Exercise 6: The altitudes of a triangle – the Euler line

Euclidean geometry:

Construct a triangle *ABC.*

Construct the three altitudes.

What characteristic property of the altitudes have you now successfully demonstrated?

Construct the circumcentre *O*, i.e. the intersection point of the perpendicular bisectors as well.

Construct the centre of gravity *G*, i.e. the intersection point of medians as well.

What can you say about the relationship between these three points: the orthocentre H (i.e. intersection point of the altitudes), the circumcentre O and the centre of gravity G?

Minkowski geometry:

Construct a triangle *ABC.*

Construct the three hyperbolic altitudes.

What characteristic property of the hyperbolic altitudes have you now successfully demonstrated?

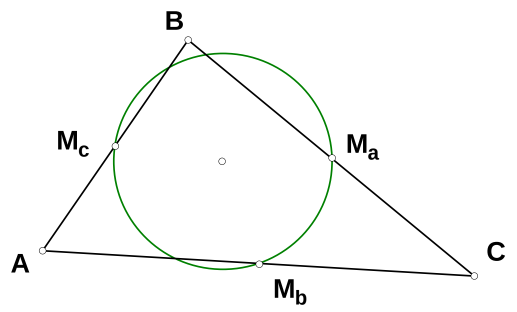
Construct the hyperbolic circumcentre *O*, i.e. the intersection point of the hyperbolic perpendicular bisectors as well.

Construct the centre of gravity *G*, i.e. the intersection point of medians as well.

What can you say about the relationship between these three points: the hyperbolic orthocentre H (i.e. intersection point of the altitudes), the hyperbolic circumcentre O and the centre of gravity G?

Exercise 7: The nine-point ‘circle’

Euclidean geometry:



Construct a triangle *ABC.*

Construct the midpoints of the three legs.

Construct the circle passing through these three midpoints.

Remark: This circle is known as the midpoint circle or the Euler circle.

It is also known as the nine-point circle and we will now try to justify this name.

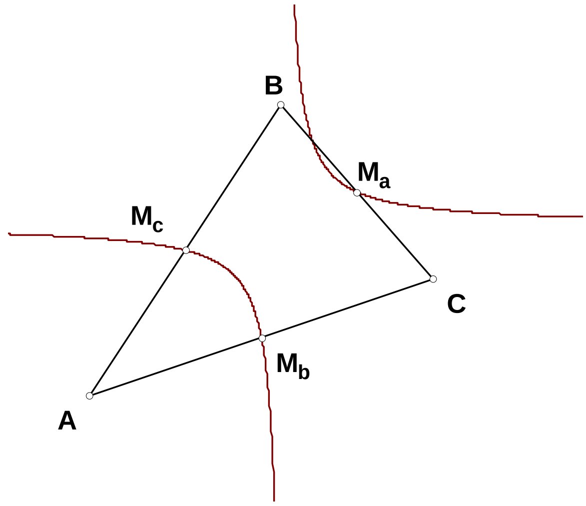
Construct the altitudes of the triangle as well.

What is the relationship between the base points of the altitudes and the midpoint circle?

The midpoint circle intersects the three altitudes in three further points. What is special about these additional intersection points?

Now you should be able to explain why it is being called a nine-point circle!

Minkowski geometry:



Construct a triangle *ABC.*

Construct the midpoints of the three legs.

Construct the rectangular hyperbola passing through these three midpoints.

Remark: This rectangular hyperbola is known as the midpoint hyperbola.

It is also known as the nine-point hyperbola and we will now try to justify this name.

Construct the hyperbolic altitudes of the triangle as well.

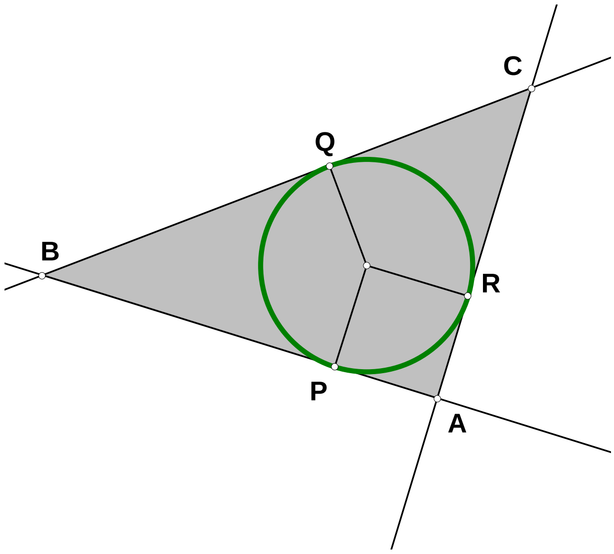
What is the relationship between the base points of the hyperbolic altitudes and the midpoint hyperbola?

The midpoint hyperbola intersects the three altitudes in three further points. What is special about these additional intersection points?

Now you should be able to explain why it is being called a nine-point hyperbola!

Exercise 8: Kissing 'circles'

Euclidean geometry:



Construct a circle and select three points *P*, *Q* and *R* on the circle.

Construct the radii and the *tangents* associated with the three points *P*, *Q* and *R*. The three tangents span a triangle *ABC* with the vertices *A*, *B* and *C* being the intersection points, such that *P* falls on the side *AB*, *Q* on the side *BC* and *R* on the side *CA*.

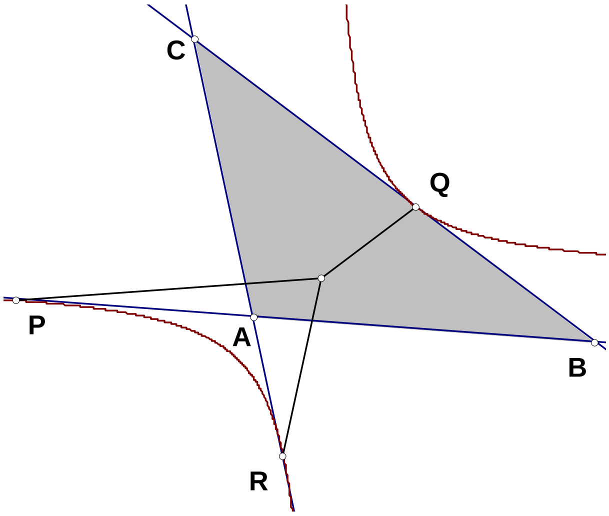
What is the relationship between the circle and the triangle?

Construct the three circles having the vertices *A*, *B* and *C* as centre and passing respectively through the three points *P*, *Q* and *R*. Hide the original circle as well as the three radii. The three remaining circles are known as *kissing circles* of the triple points *A*, *B* and *C*.

How can you construct kissing circles if the original given data is the triple *ABC*?

Will all triples of points have kissing circles?

Minkowski geometry:



Construct a rectangular hyperbola and select three points *P*, *Q* and *R* on the hyperbola.

Construct the radii and the *tangents* associated with the three points *P*, *Q* and *R*. The three tangents span a triangle *ABC* with the vertices *A*, *B* and *C* being the intersection points, such that *P* falls on the side *AB*, *Q* on the side *BC* and *R* on the side *CA*.

What is the relationship between the hyperbola and the triangle?

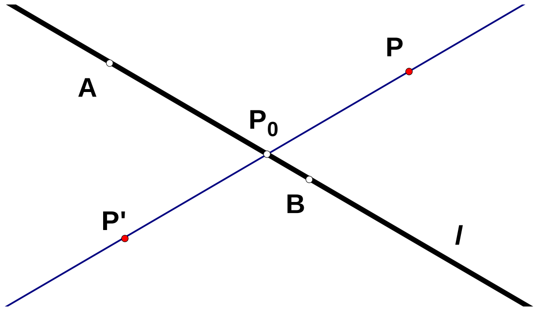
Construct the three rectangular hyperbolas having the vertices *A*, *B* and *C* as centre and passing respectively through the three points *P*, *Q* and *R*. Hide the original hyperbola as well as the three radii. The three remaining hyperbolas are known as *kissing hyperbolas* of the triple *A*, *B* and *C*.

How can you construct kissing hyperbolas if the original given data is the triple *ABC*?

Will all triples of points have kissing hyperbolas?

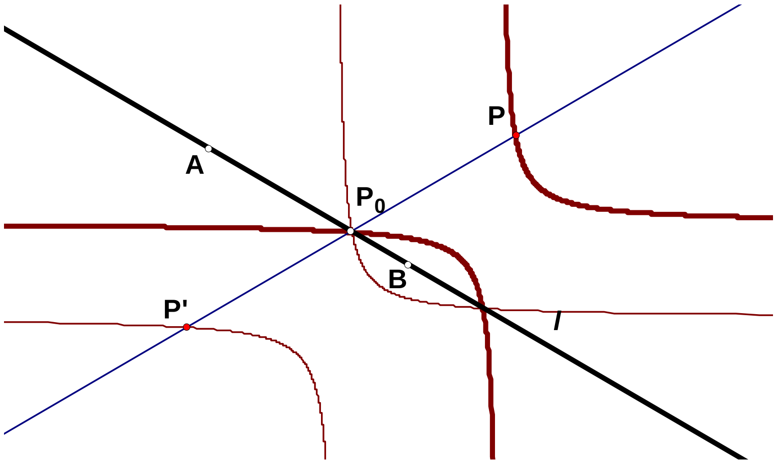
A note on Minkowski reflections

Before we continue with the next exercise it will be useful to consider transformations in Minkowski geometry in some detail. If we want to **reflect** a point *P* in the line l generated by the point *AB*, we must first construct the hyperbolic perpendicular to the mirror axis *l* through *P*:



The hyperbolic perpendicular intersects the mirror axis *l* in the foot point P_0. We then know that the reflection is the point on the perpendicular lying symmetrical around P_0. But point symmetry is common to Minkowski and Euclidean Geometry. So now we can simply rotate *P* around P_0 with 180° and the resulting point *P'* is the hyperbolic reflection.

Once you can reflect points, you can reflect other objects as well. Straight lines are reflected using two points, one conveniently being the fix point i.e. the intersection with the mirror axis. Other objects like hyperbolas are most easily reflected as loci. So you select a free point *F* on the hyperbola and reflect in the mirror axis *l*. Selecting the free point *F* as well as the reflection point *F'* you can now construct the **Locus** of *F'* driven by *F*. And that precisely is the reflected image of the original hyperbola:

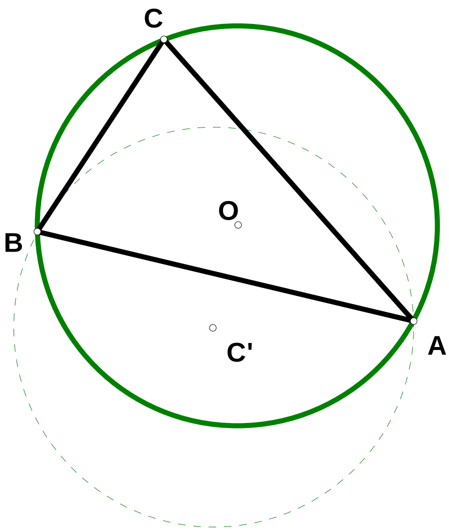


Of course the image is just another rectangular hyperbola with horizontal and vertical asymptotes. In fact the new centre is just the reflection of the old centre, so in this case it would in fact be very easy to construct the mirror image using just two points.

Remark: In fact you may even reflect a Euclidean circle in this way. A Euclidean circle in Minkowski geometry is of course just a special conic with eccentricity $\sqrt{2}$. So the resulting image is just another special conic with the same eccentricity, i.e. the image is another circle!

Exercise 9: Reflection games

Euclidean geometry:



Construct a triangle *ABC* and the circumscribed circle together with its centre the circum-centre *O*.

Reflect the circumscribed circle in the three sides of the triangle *a*, *b* and *c*. What do you observe? What is the special significance of the point associated with these three reflected circles?

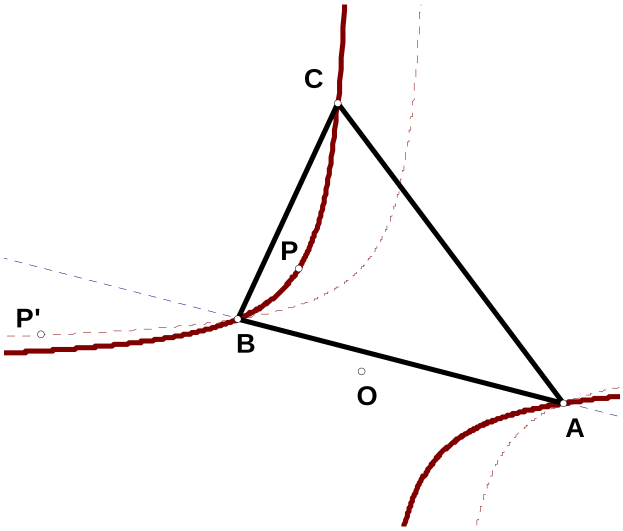
Reflect the circum-centre *O* in the three sides of the triangle thus creating the centre of the reflected circles: *A'*, *B'* and *C'*. What is the relationship between the triangle *ABC* and the dual triangle *A'B'C'*?

Repeat the construction with the dual triangle A'B'C'. What is the result?

Construct the midpoints of the original triangle as well as the midpoints of the dual triangle. What do you observe?

Construct the midpoint circle of the original triangle ABC as well as for the dual triangle *A'B'C'*. What do you observe?

Minkowski geometry:



Construct a triangle *ABC* and the circumscribed hyperbola together with its centre the circum-centre *O*.

Reflect the circumscribed hyperbola in the three sides of the triangle *a*, *b* and *c* (using hyperbolic reflections!). What do you observe? What is the special significance of the point associated with the reflected hyperbolas?

Reflect the circum-centre *O* in the three sides of the triangle thus creating the centre of the reflected hyperbolas: *A'*, *B'* and *C'*. What is the relationship between the triangle *ABC* and the dual triangle *A'B'C'*?

Repeat the construction with the dual triangle A'B'C'. What is the result?

Construct the midpoints of the original triangle as well as the midpoints of the dual triangle. What do you observe?

Construct the midpoint hyperbola of the original triangle ABC as well as for the dual triangle *A'B'C'*. What do you observe?