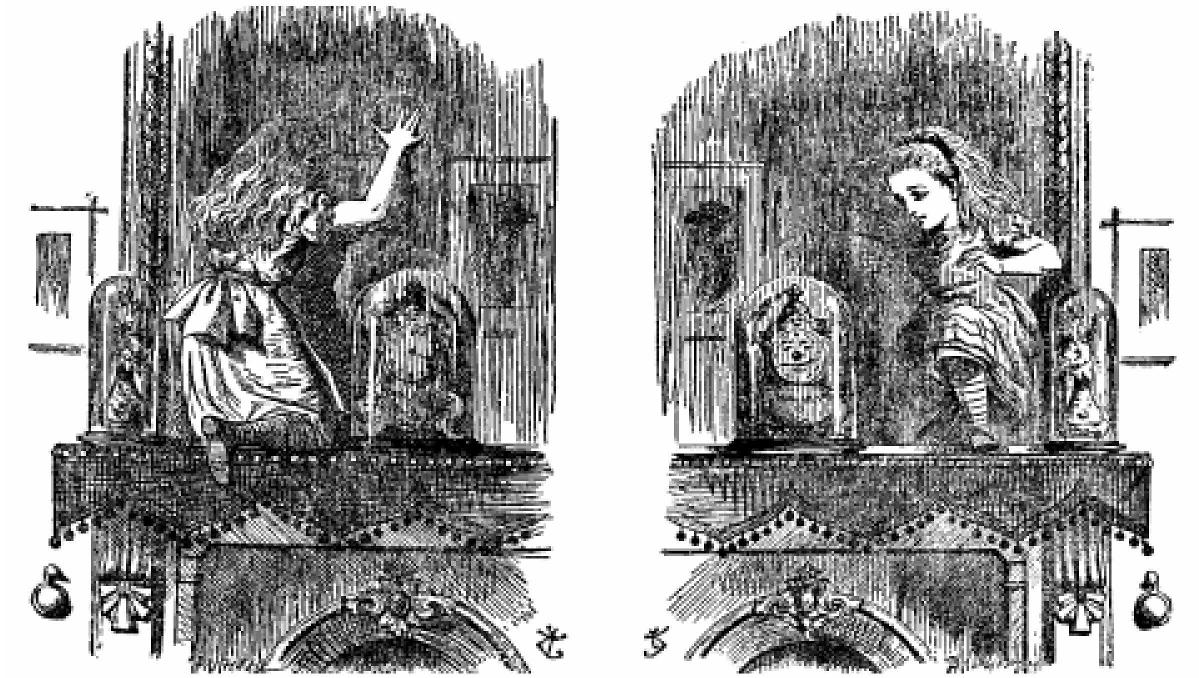
Through the Looking Glass: A glimpse of Euclid's twin geometry: The Minkowski geometry



Supplementary Notes ICME-10 Copenhagen 2004

Bjørn Felsager

Haslev Gymnasium & HF, Denmark

Preliminary remarks: Let us first remind ourselves a little about the history of the Minkowski geometry. It is a fairly new discovery going back to the beginning of the previous century, where **Einstein** in 1905 published his famous paper 'Zur Elektrodynamik bewegter Körper' which later became known as his introduction to the special relativity. In this paper he constructed a theory of space and time, where space and time no longer were considered absolute concepts, but mingled with each other in a way that could be precisely described by the so called Lorentz transformations.

The theory of relativity made an enormous impact on the contemporary scientists including many famous mathematicians. In 1907 **Minkowski** gave a most influential lecture, where he showed that the theory of special relativity could be cast into a purely geometrical theory of space and time with an invariant based upon a variant of the Pythagorean theorem:

$$d\tau^2 = dt^2 - ds^2/c^2$$

The square of the distance between two neighboring space-time events (measured in the proper time τ) is the same as the difference between the square of distance measured in the inertial (laboratory) time t and the square of the Euclidean distance s. (The velocity of light *c* takes care of the conversion between time units and space units). Minkowski concluded his lecture with the famous and dramatic prophecy:

,*Von Stund an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken, und nur noch eine art Union der beiden soll Selbständigkeit bewahren*'

Shortly afterwards in 1910 **Klein** gave an important lecture on the Minkowski geometry of space-time, where he showed how it fitted into the scheme of the Erlangen Programme: A particular kind of geometry was to be characterized by a group of symmetry transformations. The study of the particular geometry could then be considered a study of the properties of geometrical figures left invariant by the group of symmetry transformations. In the case of Minkowski geometry the group of symmetry transformations consisted of the Lorentz transformations or rather the extended group of Poincare transformations, which also included displacements.

So this is the official line of history behind the Minkowski geometry and because of it's mingling of space and time it is usually considered to be a more abstract theory than both the usual Euclidean geometry and its extensions, the spherical geometry as well as the hyperbolic geometry.

These preliminary remarks obviously raise the following question: **Why should we be interested in Minkowski geometry in this setting**? I hope to be able to produce a satisfactory answer to this question in the following discussion!

Remark: These notes have been used on several occasions as a general introduction to Minkowski geometry. In one case – at a teachers college – I was also allowed to include a workshop. I have reproduced this workshop in an accompanying paper, to show how one can supplement a general introduction with activities, that allows students to get their hands dirty in Minkowski geometry as well as letting them gain some experience themselves.

Introduction: In a high school as well as in a teachers college the courses in Euclidean geometry can be supplemented with a short excursion into non-Euclidean geometries for several reasons:

1) It is by itself a dramatic experience to realize that the Euclidean geometry is not the only possible geometry – and the discovery of the non-Euclidean geometries is certainly a very important part of the cultural history of mathematics with lots of philosophical implications.
2) Gaining experiences with non-Euclidean geometries puts Euclidean geometry itself in a new fresh perspective. You can no longer rely on your intuition and many subtleties that are easily overlooked in Euclidean geometry suddenly bring themselves to your attention.

It is for these and other reasons that short excursions into non-Euclidean geometries can be very rewarding in traditional mathematics courses at least beginning with the high school level.

But what possibilities do we then have for bringing such concepts to the student's minds? Traditionally elementary books on geometry focus exclusively upon the spherical geometries and the hyperbolic geometries. And very often a historical line of arguments is used to motivate the new geometries beginning with a discussion of the parallel axiom.

But these are not our only options! Not only do we know much more now about geometry than was known two centuries ago, when non-Euclidean geometries were first discovered – and hence we have alternative routes into non-Euclidean geometries. But in particular we now have available dynamical geometry programs, which – suitably modified – allows us to experiment and thus gain first hand experiences with non-Euclidean geometries. This has been emphasized for some time in relationship to the hyperbolic geometry, where various standard models – such as the Poincare disk model – has been successfully implemented. But is seems much less known, that it is much easier to implement tools for the Minkowski geometry than for the full hyperbolic geometry and because the Minkowski geometry is much closer related to the Euclidean geometry, it is in fact much easier to introduce.

In this lecture I will therefore outline a possible introduction to Minkowski geometry based upon the following principles:

1) The use of a dynamical geometry program such as Cabrii or SketchPad to make geometrical constructions in the Minkowski geometry immediately available to students.
2) The similarity between the usual Euclidean geometry and the Minkowski geometry is emphasized – in particular there is no mention of the space-time structure in the beginning. In stead their common ground (the affine geometry) is being exploited.
3) A dramatic setting based upon the well-known tales of Lewis Carroll – 'Alice in wonderland' and 'Through the Looking Glass' – is used to capture the imagination of the students.

Remark: Although the following content is well known, to the best of my knowledge the setting is original. In fact I don't think it will be easy to find the ideas explicitly revealed in the literature. They seem to belong to the Mathematical folklore, which are of course well known by the experts, but some how no one got the time to write them down!

So we begin with the following important questions:

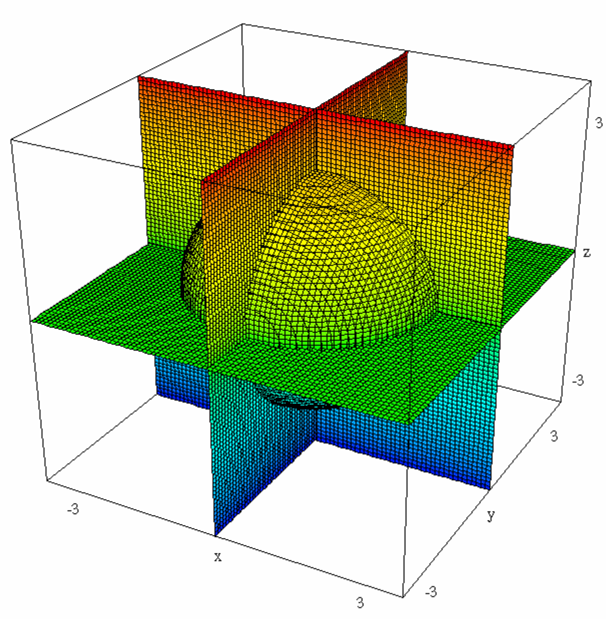
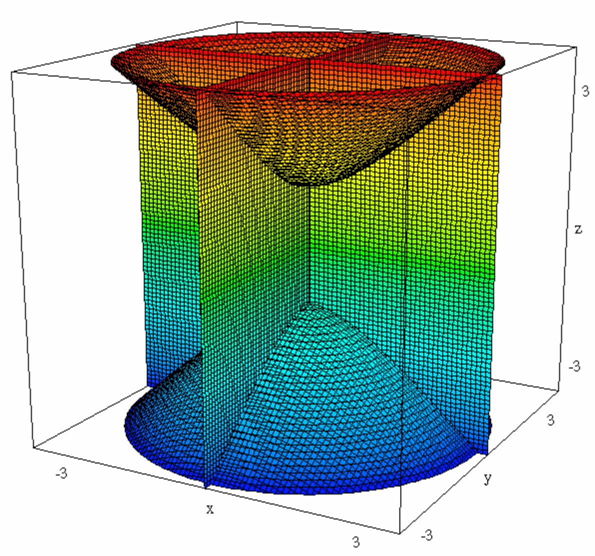
Is Euclidean geometry the only possible geometry?

Can you justify the answer in a different way from the historical line of argument (which invokes the axiom of parallel lines)?

Let's recall the axiom of parallel lines in the following version, which is particularly simple and in particular it does not involve any advanced concepts such as circles or angles:

Playfair's axiom:
Through a point not on a given line there is one and only one line parallel to the given line (i.e. that does not intersect the given line).

We begin by revealing that there are four simple geometrical structures you can put upon a two-dimensional space, two of which obeys the parallel axiom (i.e. the corresponding space is flat) and two of which fails to obey the parallel axiom (i.e. the corresponding space is curved):

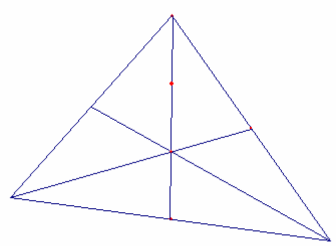
Euclidean geometry (2d): Geometry of straight lines and circles. Circular trigonometry: sine and cosine	**Minkowski geometry (2d):** Geometry of straight lines and rectangular hyperbolas. Hyperbolic trigonometry: exp and ln
Spherical geometry:  Geometry of a spherical surface in Euclidean space. Spherical trigonometry.	**Hyperbolic geometry:**  Geometry of a hyperbolic surface in Minkowski space. Hyperbolic trigonometry.

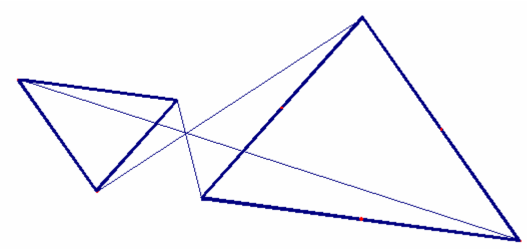
So the Euclidean geometry does posses a twin, the Minkowski geometry, which avoids curvature and hence satisfies the axiom of parallel lines. The two geometries share the affine structure of plane, i.e. they share:

- Points
- Straight lines
- Parallel lines

- Ratios
- Midpoints
- Medians

- Translations
- Multiplications
- Similar figures

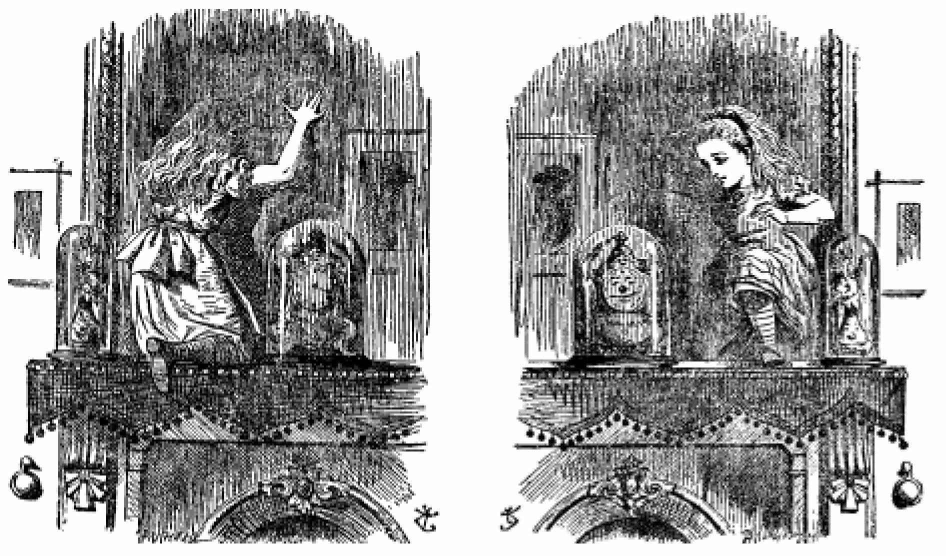


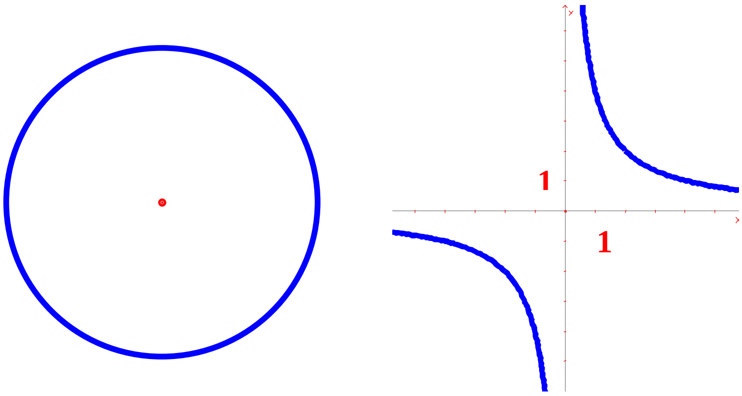


But is it also possible to give an *elementary* introduction to the Minkowski geometry avoiding abstract concepts such as the geometrical structure of space-time?

We suggest the following strategy: A Euclidean geometry is based upon a view on symmetry that makes the circle the most symmetrical figure – since ancient times considered the most perfect figure. This view e.g. dominated astronomy for several thousands years. But is it possible to imagine another view of the world, where it is not the circle, but the rectangular hyperbola, that is considered the most symmetrical figure? To make the transition to this alternative view more potent, we imagine that we follow Alice through the looking glass, and that she precisely discovers it is the alternative view, that prevails behind the looking glass:

Following Alice through the Looking Glass:





The most symmetrical figure as seen
in front of the looking glass – behind the looking glass

The adoption of the rectangular hyperbola as the most symmetrical figure requires the introduction of new structures in Minkowski geometry:

- **Two asymptotic directions**: Vertical/ horizontal
- **A different concept of right angles:** We need to replace the traditional right angles, because of their intimate relationship with the geometry of the circle!

Right angles in the Minkowski geometry: To motivate the introduction of hyperbolic right angles we need some characteristic properties of a right angle, which links the right angle to a circle. Which property does not matter! We chose the following one, since this is a very basic one, easy to understand also on an intuitive level.

- **The tangent is perpendicular to the radius**

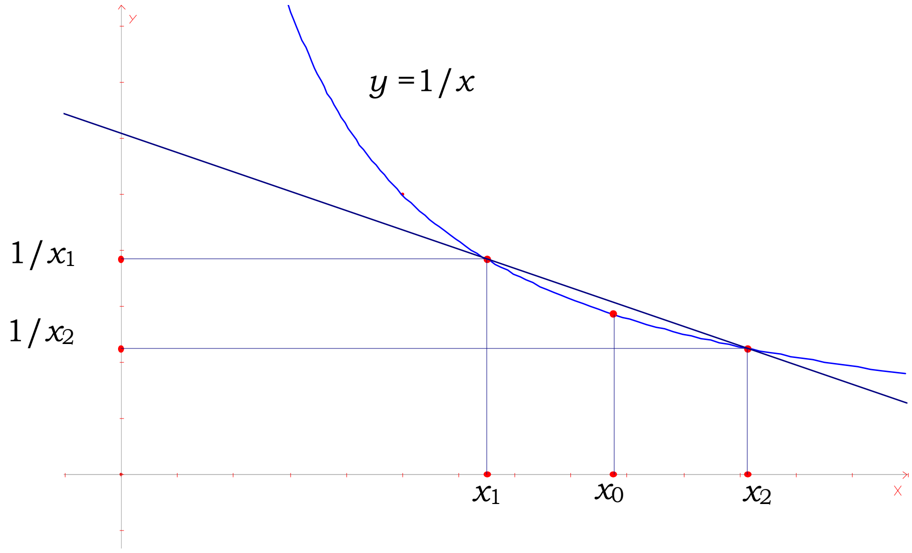
To understand how tangents of a hyperbola behave we will use some analytical geometry. You may find this a little disturbing: Why not use elementary geometry all the way through. There are two reasons. The first one is that analytical geometry is in fact much simpler in relation to the Minkowski geometry than the Euclidean geometry. The second is that we lack intuition about the structure of figures in Minkowski geometry. For these reasons simple proofs in Minkowski geometry tends to be easier to follow using a little analytical geometry!

As a starting point we therefore take the following observation about the slope of a secant. By the way this is the only detailed argument based upon analytical geometry I will present in this paper. There will thus be lots of opportunities for the reader verifying results analytically later on your own!

The theorem of the secant: To determine the slope of a secant we perform the following standard calculation:

$$\alpha_{sekant} = \frac{\frac{1}{x_2} - \frac{1}{x_1}}{x_2 - x_1} = \frac{\frac{x_1 - x_2}{x_2 \cdot x_1}}{x_2 - x_1} = -\frac{1}{x_2 \cdot x_1}$$

(see the figure attached!)



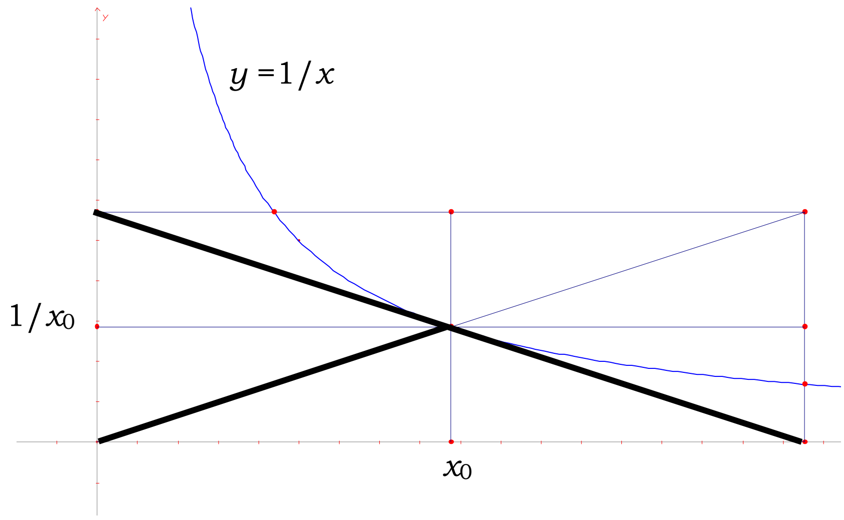
Notice in particular that the slope of the secant only depends upon the **product** of the abscissa x! This has deep implications for the addition of hyperbolic angles!

To compute the slope of the tangent, we now simply let the two endpoints of the secant coincide. We thereby find the following well-known result:

$$\alpha_{\text{tangent}} = -\frac{1}{x_0^{\,2}}$$

Notice that we have managed to derive this elementary result without appealing to calculus! Next we compare the slope of the tangent to the slope of the radius:

$$\alpha_{\text{tangent}} = -\frac{1}{x_0^{\,2}} \quad \text{and} \quad \alpha_{radius} = \frac{\frac{1}{x_0}}{x_0} = \frac{1}{x_0^{\,2}}$$

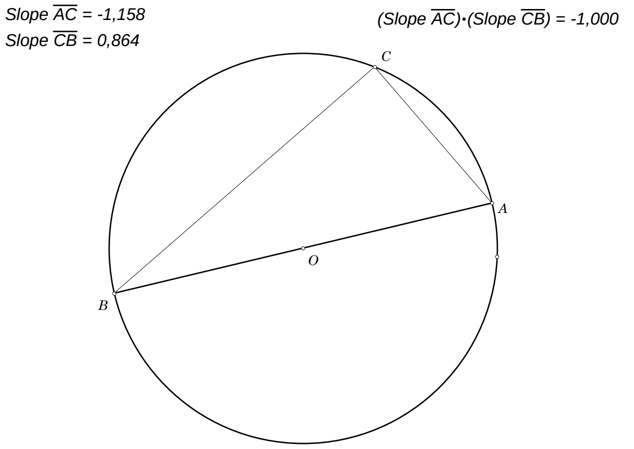


Conclusion: In Minkowski geometry two lines are perpendicular precisely when they have opposite slopes!

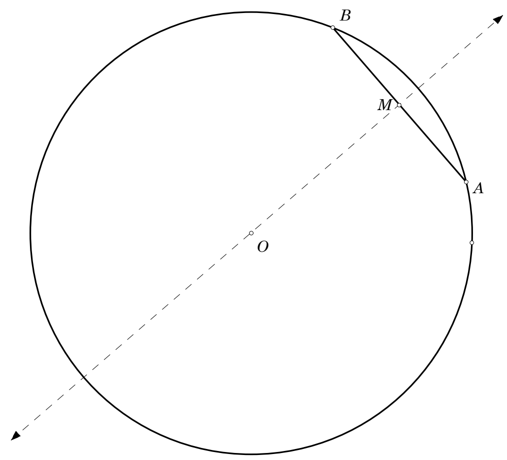
This observation is the main result. In particular it makes it easy to investigate simple properties of right angles!

Recall the following theorems linking ordinary right angles to circles:

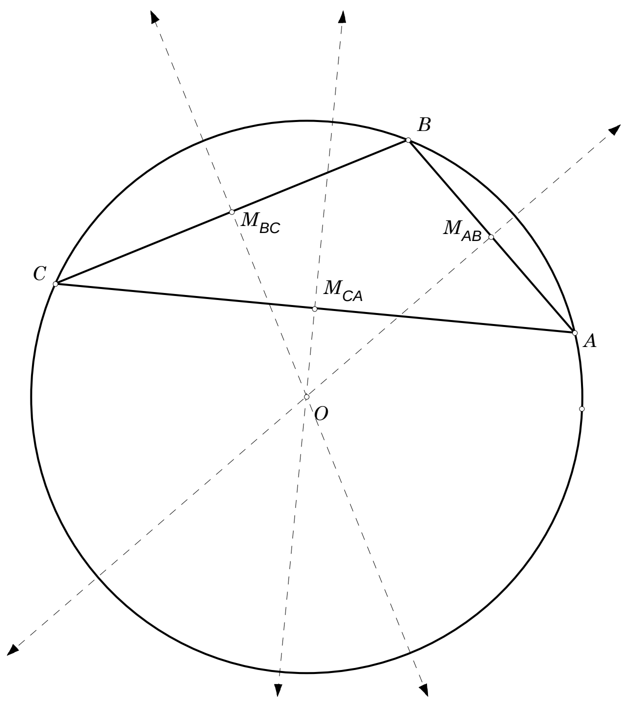
Example 1 – First theorem of circles: **Thales' theorem:**
An angle is a right angle precisely when it spans the diameter.



Example 2 – Second theorem of circles: **The theorem of Chords:**
The perpendicular bisector of a chord passes through the center.



Example 3 – Third theorem of circles: **The circumscribed circle of a triangle**:
The perpendicular bisectors of a triangle pass through the same point, the center of the circumscribed circle (i.e. the circumcentre O).

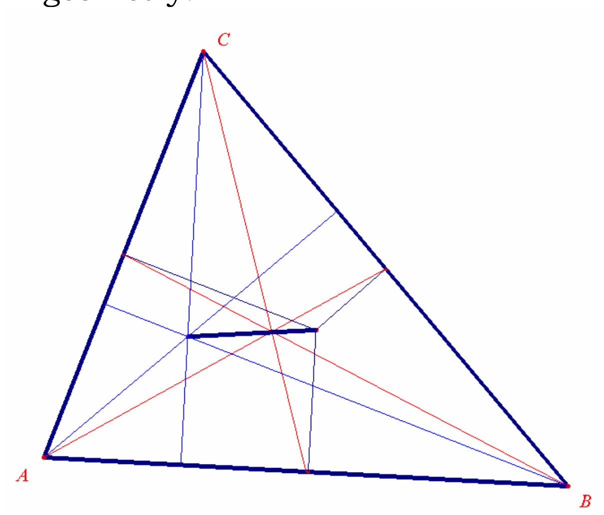


In each of the above cases it is elementary to verify the corresponding theorem in Minkowski geometry experimentally using a Dynamic Geometry program capable of drawing rectangular hyperbolas (a hyperbolic compass!). We leave these verifications as exercises (see the accompanying workshop).

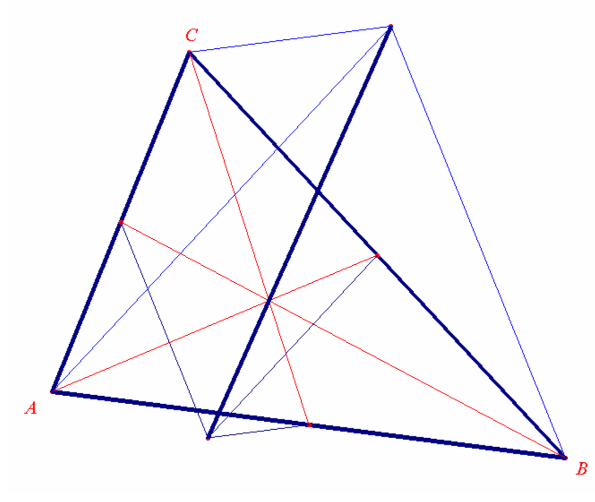
Other more non-trivial examples of correspondences between the Euclidean geometry and the Minkowski geometry involve the Euler line and the nine-point circle:

The Euler line:

a) **The Euler line** in Euclidean geometry:

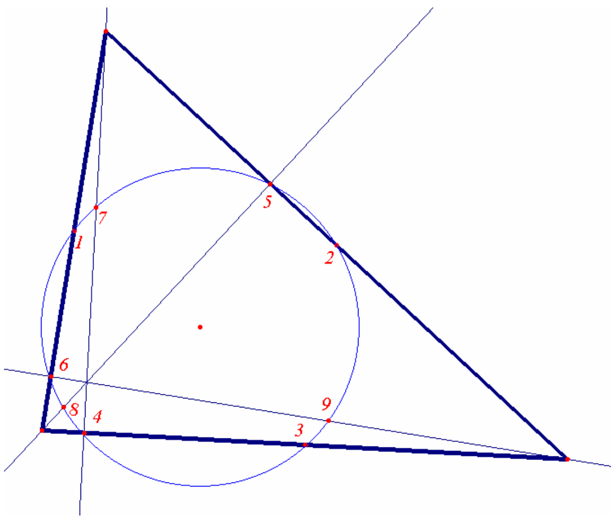


b) **The Euler line** in the Minkowski geometry:

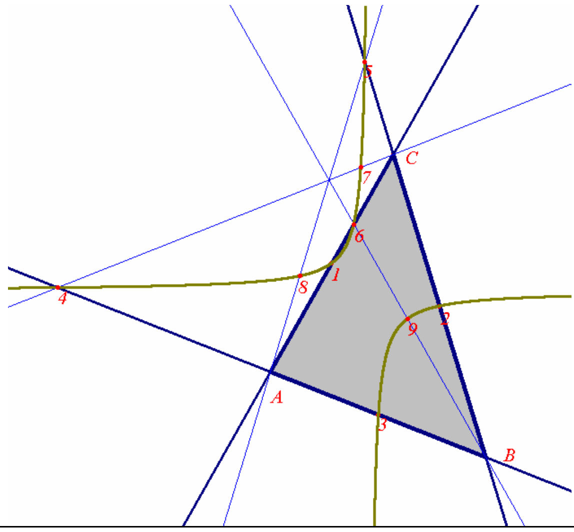


The nine-point configuration:

a) **The nine-point circle**:



b) **The nine-point hyperbola**:

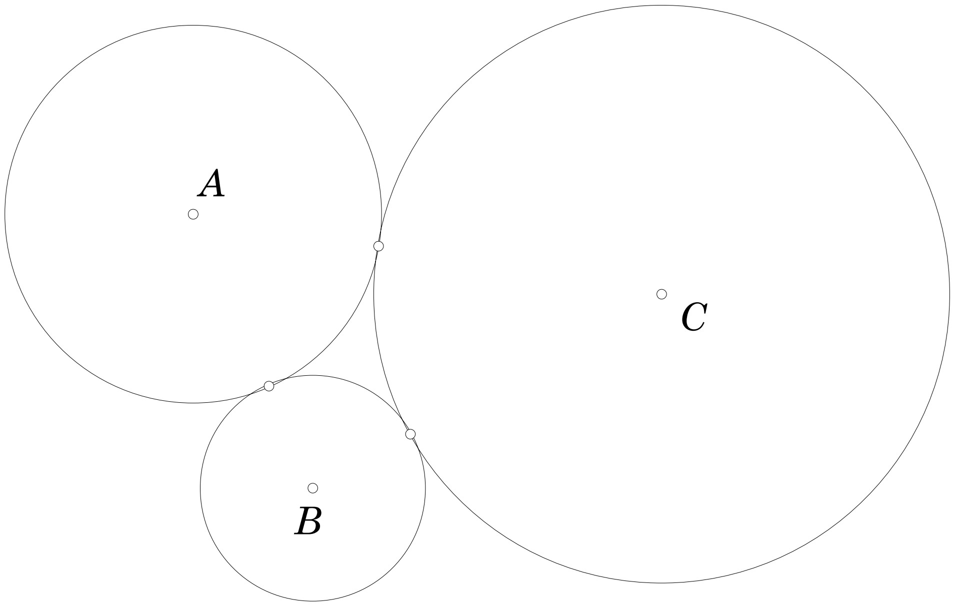


Notice that both cases are special cases of a theorem in affine geometry, which says that the heights in the triangle can be replaced by any set of three lines from the vertices passing through the same point. This gives rise to nine points through which a unique conic passes, the **nine-point conic**.

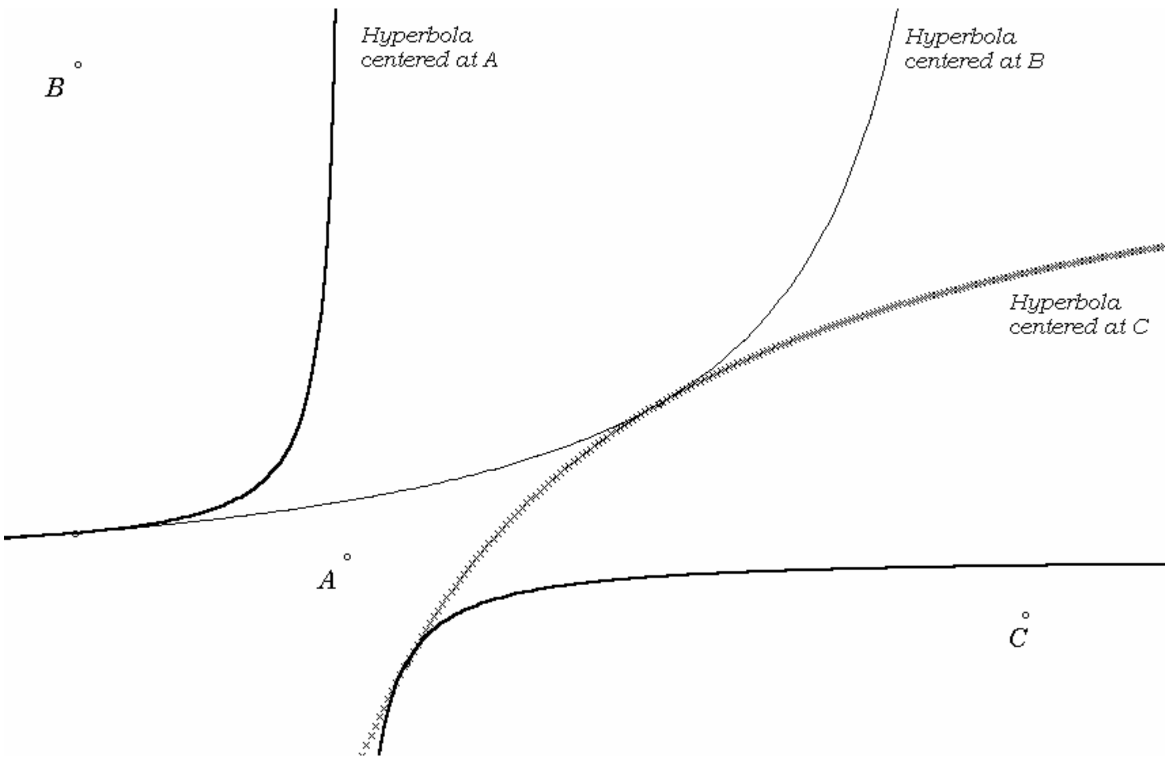
- In the case of an ellipse, you may consider the configuration to be a parallel projection of the nine-point circle.
- In the case of a hyperbola, you may similarly consider the configuration to be a parallel projection of the nine-point hyperbola.

In both cases the configuration is a **shadow** of a corresponding simpler configuration in Euclidean respectively Minkowski geometry. The above configuration with a nine-point conic is thus one of the places, where two separate theorems from Euclidean and Minkowski geometry are unified in affine geometry!

As a final example of such a correspondence we will look at the **kissing circles**. In Euclidean geometry any triple of points admits circles, which kiss each other (tangentially) e.g. as shown on the following figure:



In Minkowski geometry it is slightly more complicated, but in many cases you can still find kissing hyperbolas as shown on the following figure:



Important remark: At this point you may perhaps think that all results in Euclidean geometries involving circles have a Minkowski counterpart. But that is not the case: The symmetry structure of the two geometries also has characteristic differences. E.g. rotations in Euclidean space have a repetitive periodic structure unlike the rotations in Minkowski space, where the asymptotic directions break the periodicity. As a consequence the Minkowski geometry lack regular polygons. And thus the regular polygons constitute an example of an important concept in Euclidean geometry, which has no correspondence in Minkowski geometry. But for pedagogical reasons we have emphasized the striking similarities rather than the (also important!) differences between the two twin geometries.

At this point you should now have obtained some feeling for the Minkowski geometry and we proceed with a discussion of the most basic theorem in Minkowski geometry – the analogue of the Pythagorean theorem, which controls all distance calculations!

We present the derivation in the form a dialogue between Alice, the Mock Turtle and the Gryphon starting with a famous dialogue written by Lewis Carroll for 'Alice in Wonderland':

'I couldn't afford to learn it,' said the Mock Turtle with a sigh. 'I only took the regular course.'

'What was that?' inquired Alice.

'Reeling and Writhing, of course, to begin with,' the Mock Turtle replied; 'and then the different branches of Arithmetic – Ambition, Distraction, Uglification, and Derision.'

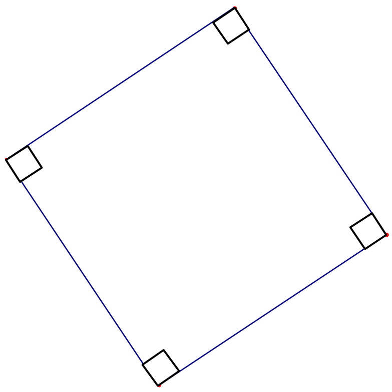
'I never heard of "Uglification",' Alice ventured to say. 'What is it?'

The Gryphon lifted up both its paws in surprise. 'Never heard of uglifying!' it exclaimed. 'You know what to beautify is, I suppose?'

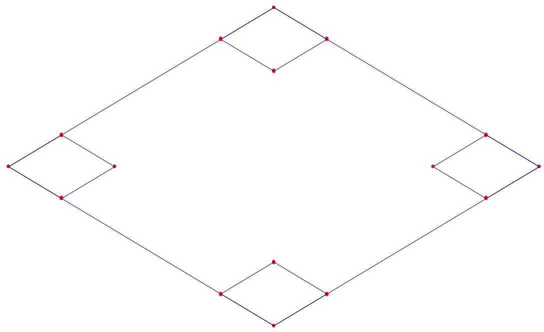
A fictitious dialogue between Alice and the Gryphon about geometry:

'I do suppose you know what a square is?' the Gryphon exclaimed.
'Of course' Alice replied. ' It's a totally symmetrical quadrilateral with four right angles'.
And to prove that she really understood what she was talking about, she made a sketch of a square:

Alice makes a drawing of a square:



The Gryphon makes a drawing of a square:



Alice claims the Gryphon's square is a **diamond** (i.e. a rhombus with horizontal and vertical diagonals)!

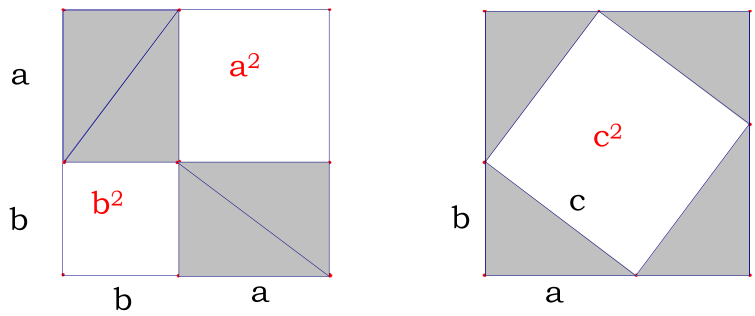
'Oh no', the Gryphon said in surprise: 'That's not a square – It's just some silly parallelogram! This is how a square looks like!'
To Alice surprise the Gryphon made a sketch of a diamond figure. 'Is that what a square looks like?' she exclaimed.
'Of course! Every child knows that a square has four right angles and is totally symmetrical! Don't you learn anything in your schools? Didn't they ever tell you about the Pythagorean theorem?'
'Yes they did', Alice replied cautiously, ' The Square of the hypotenuse is the sum of the Squares of the legs'.
'What are you talking about', the Gryphon replied, not believing what it just heard: 'Every child knows that the square of the hypotenuse is the ***difference*** *between the square of the legs'.*
'But I thought I had a proof?' Alice dared to say.
'Proof', snorted the Gryphon. 'You don't even know what a square is!'

And the conversation continues with Alice demonstrating her proof and the Gryphon demonstrating his proof. Both use the same simple argument: First they decompose a square according to the formula for the square of a binomial:

$$(a+b)^2 = a^2 + b^2 + 2ab$$

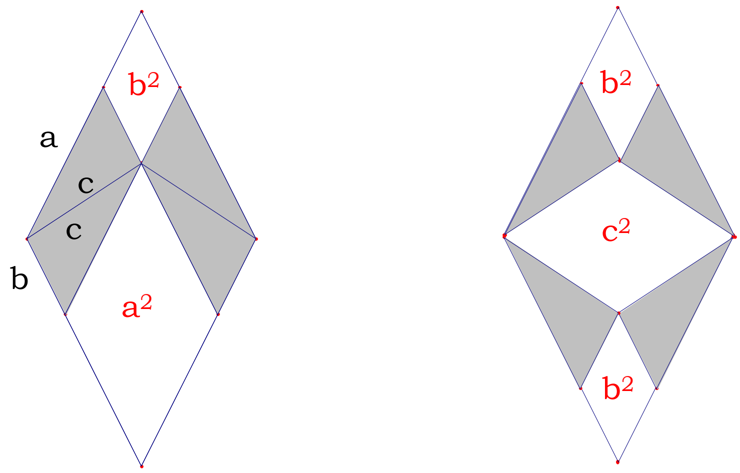
Next they rearrange the figure suitably and the Pythagorean theorem follows immediately by comparing the two figures obtained in this way and ignoring the common right angled triangles:

Alice explains the Pythagorean theorem:



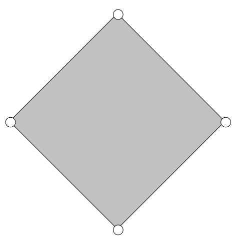
$$\boldsymbol{c^2 = a^2 + b^2}$$

The Gryphon explains the Pythagorean theorem:



$$b^2 + a^2 = 2b^2 + c^2 \;\Rightarrow\; \boldsymbol{c^2 = a^2 - b^2}$$

Remark: Once we have established the Pythagorean theorem for the Minkowski geometry we can make some important observations. There exist a Euclidean square, which is also a Minkowski square, namely the square with slopes ± 1:



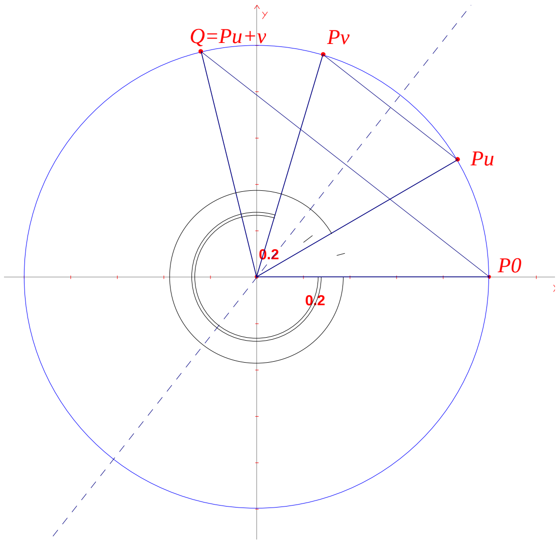
It is then obvious to assign them this common square the same length of the side in the two geometries. Consequently the common square also gets the same area in the two geometries. Since collages of such special squares can approximate any simple figure, it follows that the area of any simple figure must in fact be the same in the two geometries. Thus they also have areas in common!

It is now easy to show that a rectangular hyperbola can be characterized as a set of points with constant hyperbolic distance to a center etc.

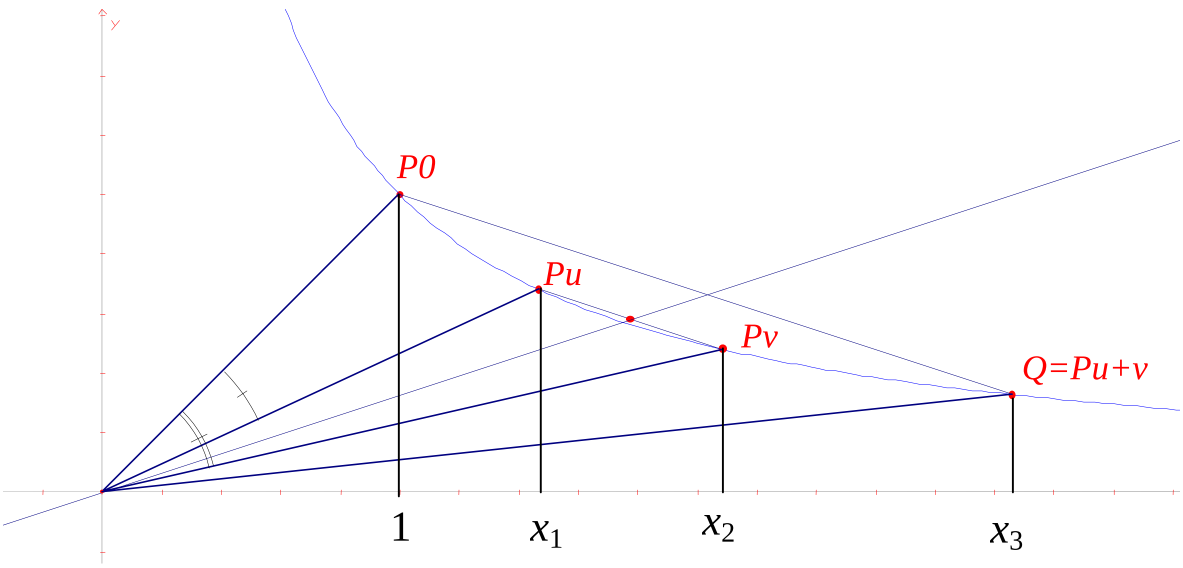
Once we control distances we can also introduce trigonometry. In fact the so-called hyperbolic trigonometry is precisely the trigonometry associated with Minkowski geometry! As a preparation for trigonometry we must first introduce a measure for the hyperbolic angles (as opposed to the usual circular measure of angles). The starting point is a very important remark concerning addition of angles:

On addition of angles:

a) Adding circular angles: P_0P_{u+v} is parallel to P_uP_v:



b) Adding hyperbolic angles: P_0P_{u+v} is parallel to P_uP_v:



The theorem of secants has the following important consequence:

The slope of P_0Q: $-\dfrac{1}{1\cdot x_3}$ The slope of P_uP_v: $-\dfrac{1}{x_1\cdot x_2}$

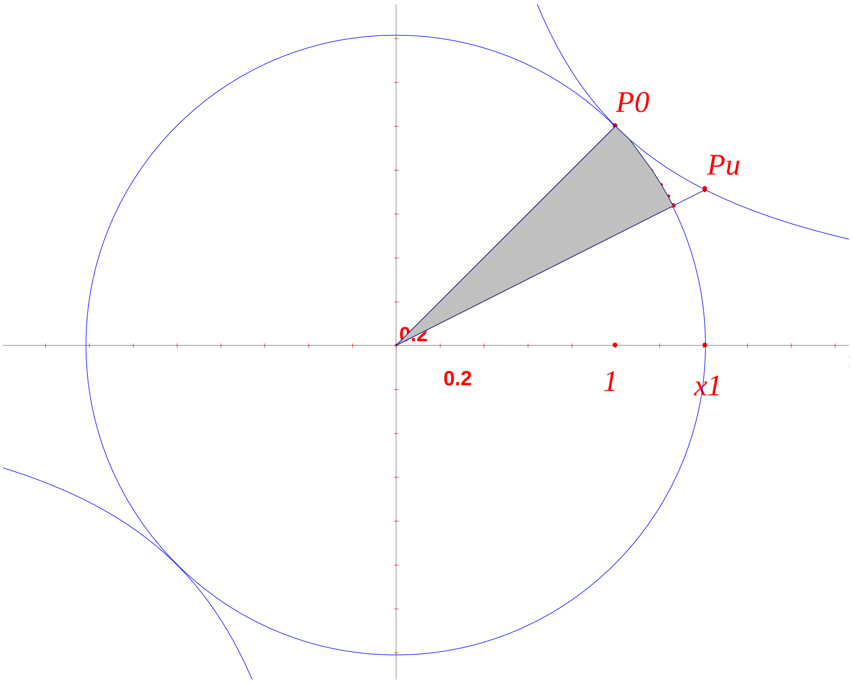
Conclusion: $x_3 = x_1 \cdot x_2$

The **hyperbolic angle** is thus a logarithmic function of the associated abscissa since we get the following identity for the measure hyp(x) of the hyperbolic angle:

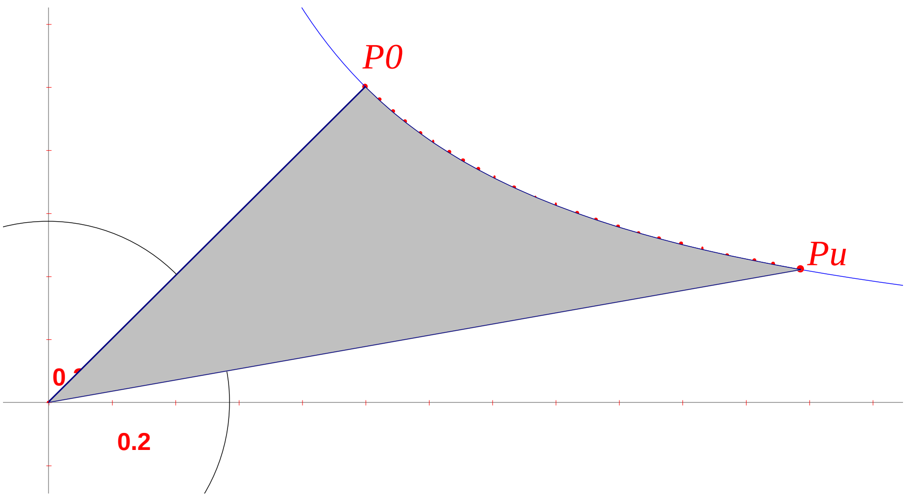
$$u + v = \text{hyp}(x_1) + \text{hyp}(x_2) = \text{hyp}(x_3) = \text{hyp}(x_1 \cdot x_2)$$

This makes it obvious to identify the measure of hyperbolic angle with the area of the sector OP_0P_u, which makes sence, since on the one hand it is a well known fact – a fact that is elementary to verify! – that the area is also a logarithm function of the associated abscissa. On the other hand the circular angle is represented by an area in Euclidean geometry:

Angles in Euclidean geometry (notice that the circle goes through the 'unit-point' (1,1) and that the circle has the total area 2π, since the radius is √2!):



Angles in Minkowski geometry: u = hyp(x) = Area(sector OP_0Pu).



Conclusion: Hyperbolic angles generate **natural logarithms**: u = ln(x)

Notice that the hyperbolic measure of angles leads to a very simple canonical parametrization of the rectangular unit hyperbola, $x{\cdot}y = 1$, in terms of the anglular measure (hyperbolic 'radians'). Since u = ln(x), we immediately get the abscissa expressed through an inverse natural logarithm, i.e. a natural exponential function: $x = \exp(u)$. The ordinate is the reciprocal value, i.e. $y = \exp(-u)$. In contrast to ordinary trigonometry, where it is customary to introduce two trigonometric functions cosine and sine, we thus need only one basic trigonometric function for for the hyperbolic trigonometry: exp.

Canonical parametrizations
in Euclidean geometry and Minkowski geometry:

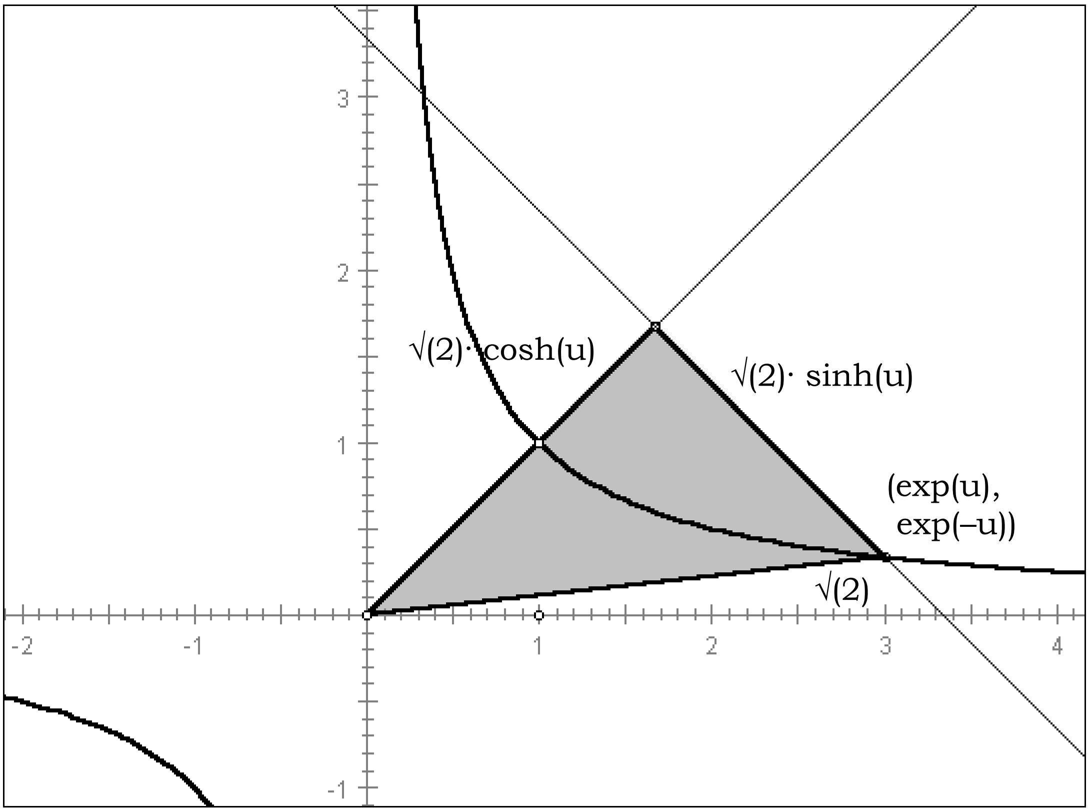
a) The unit circle:

$(x, y) = (\cos(u), \sin(u))$

b) The unit hyperbola:

$(x, y) = (\exp(u), \exp(-u))$

This makes it possible to introduce hyperbolic trigonometry in precisely the same way you introduce circular trigonometry using right angled triangles:



$$\sinh(v) = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{\exp(u) - \exp(-u)}{2}$$

$$\cosh(v) = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{\exp(u) + \exp(-u)}{2}$$

$$\tanh(v) = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{\exp(u) - \exp(-u)}{\exp(u) + \exp(-u)}$$

We leave the details as an exercise!